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A theoretical study on distortion induced fatigue of slender web curved I-girders subjected to pure bending

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Abstract
The slender web of I-girder bridges subjected to pure bending undergoes large deformations at the compressive region. This can occur at load levels less than the theoretical bend buckling limit due to the presence of initial geometric imperfections. Secondary bending stresses amplify the web stress upon receiving additional loads, i.e., traffic load. While the total web stress does not affect the resistance significantly, the cyclic component of the stress can lead to fatigue cracks at the web-to-flange connection area. The phenomenon, which is referred to as web breathing, has been studied comprehensively for straight girder. However, there is a lack of knowledge on the breathing of curved web panels. The non-colinear internal forces of curved girders causes web lateral distortions and high membrane stresses that amplifies the web breathing compared to straight girders. Theoretical studies associated with the curved web can be divided into two categories. First, very limited research investigated the curvature effect at web panel boundaries that is critical for the fatigue limit state. The simplified methods, based on beam theory, are not capable of properly modeling the actual behavior of curved girders having slender webs. Second, the more accurate analytical models, based on plate theory, were conducted for establishing the ultimate web stress and lateral displacements corresponding to strength and stability limit state, respectively. This paper reviews the analytical approaches related to the breathing of straight girders and presents a theoretical method for defining the web stresses at the web panel boundaries under pure bending moment.

1. Introduction

There is a limited understanding of curved web panels for fatigue considerations compared to the stability and strength design criteria (Linzell et al. 2004). Only two studies, one in the U.S and one in Japan, in 1980 and 1990, respectively, researched fatigue of curved girders (M. Jalali et al. 2020). In the absence of sufficient experimental studies, analytical methods assist in better understanding the fatigue mechanisms specific to curved web panels. This article presents

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ongoing research on the analytical study of curved web panels under bending moment for fatigue limit.

2. Flat web panel

When a plate girder is loaded by repeated bending, three types of fatigue cracks appear at the web panel boundary elements, i.e., flanges, and stiffeners, as shown in Fig. 1. Crack type 2 and 3 resulted from in-plane membrane and bending stress, respectively. Initial imperfections and high load levels cause the compression region to bulge out of the plane and forms crack type 1 at the web-to-compression flange location. This phenomenon is referred to as web breathing (Roberts and Davies 2002). Crack type 1 is caused by large web surface stress normal to flange, denoted by $\sigma_{\perp}$, consisting of web membrane stress in the transverse direction and out-of-plane bending stress. For pure bend loading, detailed measurements of test girders (Kuhlmann and Günther 1999) showed that the secondary bending stress σ_b is much larger than the membrane stress in the transverse direction and is almost equal to the $\sigma_{\perp}$.

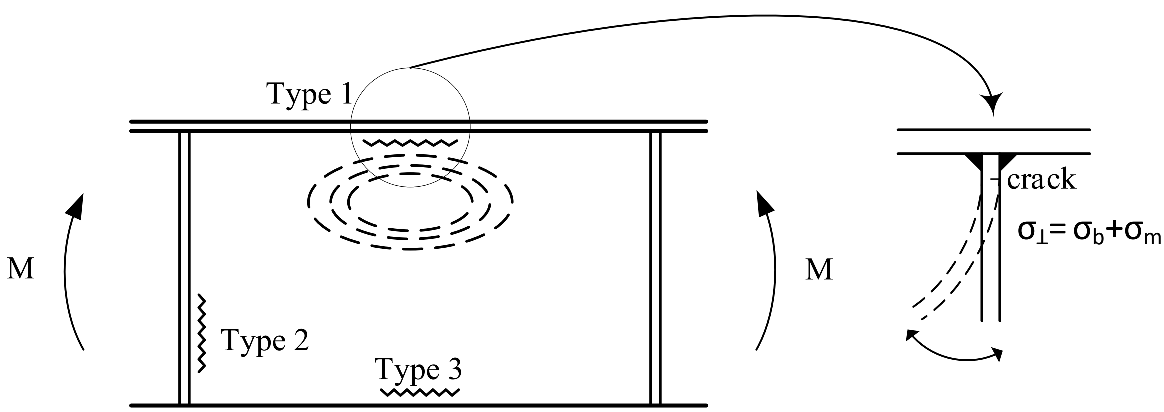

Figure 1. web breathing

Web breathing is a complex problem that corresponds to the post-buckling state and requires advanced solutions to define the non-linear web response and secondary bending stresses. Marguerre (1938) developed the governing differential equation (G.D.E) for a plate with initial deformation:

$$D_p \nabla^4 w = t\left[\frac{\partial^2 \Phi}{\partial y^2}\frac{\partial^2 (w+w_0)}{\partial x^2} + \frac{\partial^2 \Phi}{\partial x^2}\frac{\partial^2 (w+w_0)}{\partial y^2} - 2\frac{\partial^2 \Phi}{\partial x \partial y}\frac{\partial^2 (w+w_0)}{\partial x \partial y}\right] \tag{1}$$

$$\nabla^4 \Phi = E\left[\left(\frac{\partial (w+w_0)^2}{\partial x \partial y}\right)^2 - \frac{\partial^2 (w+w_0)}{\partial x^2}\frac{\partial^2 (w+w_0)}{\partial y^2} - \left(\frac{\partial^2 w_0}{\partial x \partial y}\right)^2 + \frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 w_0}{\partial y^2}\right] \tag{2}$$

where $D_p = \frac{Et^3}{12(1-\nu^2)}$ is the plate bending stiffness, E is the elastic modulus, t is the plate thickness, ν is the Poisson's ratio, ∇ is the Del operator, w is the plate deflection function, w_0 is the initial deformation, Φ is the Airy stress function.

Eq. 1 and 2 is a coupled nonlinear biharmonic differential equation without closed-form solutions, and numerical methods such as Galerkin approximate the solution space. The secondary bending stresses at the web panel boundaries can be found in terms of the plate deflection function w. Analytical approaches, including the simplified methods and efforts to solve the Marguerre equation for defining the secondary bending stresses at the web panel boundaries, are discussed in the following.

Muller and Yen (1966,1968) applied a semi-empirical method to define the web panel secondary stresses by fitting a fourth-order polynomial to the measured web lateral deformations. The stiffener and flange rigidities were considered through compatibility equations in terms of web deflections. Further tests (Patterson et al. 1970) verified their proposed approach, and the following web slenderness requirements for fatigue limit were reflected in the AASHTO (1989):

$$\beta \le \frac{3047}{\sqrt{F_y}} \text{, for girders without longitudinal stiffener} \quad (3)$$

$$\beta \le \frac{6094}{\sqrt{F_y}} \text{, for girders with longitudinal stiffener} \quad (4)$$

where β is the web thickness to web height ratio, and F_y is the yield stress in Mpa.

Maeda and Okura (1984) attempted to solve the Marguerre equations by utilizing the Galerkin method for the web panel shown in Fig. 2. The assumed initial deformation and plate deflection response were given by:

$$w_0(x,y) = e_0 \sin\frac{\pi x}{a} \sin\frac{n\pi y}{b} \quad (5)$$

$$w\,(x,y) = \sin\frac{\pi x}{a}\left(e_1 \sin\frac{\pi y}{b} + e_2 \sin\frac{2\pi y}{b}\right) \quad (6)$$

where w_0 is the initial deformation, w is the deflection function, and $e_{i's}$ are the coefficients. Given that the Sine function cannot model the tangency condition at the fixed horizontal edges, the Galerkin method could not be applied entirely. Consequently, the analytical model only served as a mathematical formulation, and the coefficients were found by finite element analysis. The solution resulted in two dependent expressions that define the displacement at point B (a/2,b/4) and secondary bending stress at point A (a/2,0). Based on experimental results (Maeda et al. 1976) and regardless of the analytical approach, it was proposed that crack Type 1 due to web breathing does not occur if crack Type 2 is prevented.

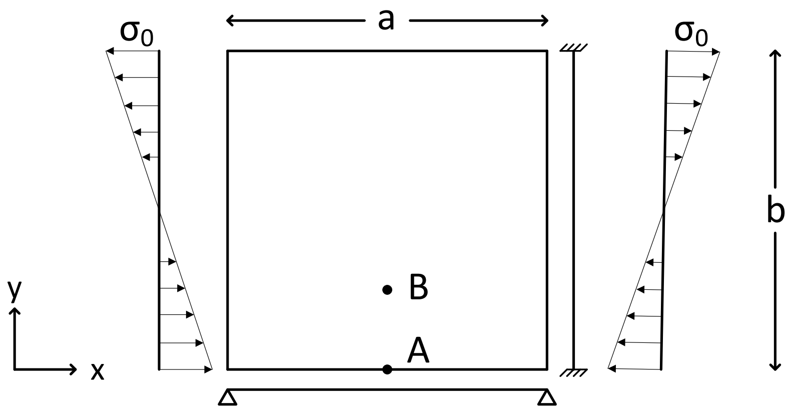

Figure 2. Maeda et al. plate loading and boundary condition

Dubas (1992) modeled the compressive region of the web panel by the simple framework made of cross beams in the two directions. The same limit as Maeda and Okura (1984) was concluded. Remadi et al. (1995) further improved the analytical model of Maeda and Okura (1984) by considering larger aspect ratios, additional deflection function coefficients, and a higher number of half-waves. The following limit for prevention of crack Type 1 based on parametric FEM analysis was proposed (Aribert et al. 1996):

$$\max\ \sigma_0 \le \min\left\{\frac{71}{1-ST}\left(\frac{N}{mm^2}\right), \sigma_{cr,F-S}\right\} \tag{7}$$

where σ_0 is the in-plane stress, ST is the stress ratio, and $\sigma_{cr,F-S}$ is the linear elastic buckling stress of the plate with fixed and simply supported longitudinal and vertical edges, respectively.

Spiegelhalder (2000) solved the Marguerre equation for web plates under bending with all edges fixed and considered the flange stiffness in terms of additional Airy stress functions. A parametric FEM model calibrated with the test data was used for developing a design limit instead of the pure analytical solution of the Marguerre equation, due to the extreme computational cost of the mathematical formulation,

$$\max \sigma_0 \le \min\left\{\frac{\Delta\sigma_c^{Type2}}{1-ST}, 2\sigma_{cr}\right\} \tag{8}$$

where $\Delta\sigma_c^{Type2}$, is the fatigue strength of crack type 2, σ_{cr} is linear elastic buckling of the plate with all edges simply supported.

3. Curved web panel

The elastic behavior of curved web panels has been studied extensively through purely theoretical analysis and numerical methods. The main focus of the works was to assist in understanding the geometric nonlinear effects and developing web bending stresses limits in terms of moment reduction factors with respect to straight web panels. In addition, setting a limit on web out-of-plane displacement was investigated for stability concerns. Featured analytical researches resulted in defining design guideline limits or highlighting important characteristics of curved web panels behavior are presented in the following.

Dabrowski (1968) developed the nonlinear G.D.E of curved web panels under pure bending based on shell theory that assisted in the fundamental understanding of curved web panels behavior:

$$D_p \nabla^4 w = t\sigma_y^0 \left(\frac{1}{R} + \frac{\partial^2 w}{\partial y^2} \right) + \frac{t}{R}\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 \Phi}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 \Phi}{\partial y^2}\frac{\partial^2 w}{\partial x^2} \tag{9}$$

$$\frac{1}{Et}\nabla^4 \Phi = -\frac{1}{R}\frac{\partial^2 w}{\partial x^2} + \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \tag{10}$$

where $D_p = \dfrac{Et^3}{12(1-\nu^2)}$ is the plate bending stiffness, σ_y^0 is the maximum normal stress in the y direction, and R is the panel radius of curvature. The solution of Dabrowski equations revealed that web membrane stress distribution deviates from linear beam theory at high loads (Wachowiak 1967). Washizu (1975) derived another form of nonlinear G.D.E of curved web panel for additional loading conditions based on the total potential energy.

Culver et al. (1972) investigated the maximum web bending stress by dividing the web panel into cylindrical strips supported elastically, i.e, spring foundation. The stiffness of each strip was calculated by considering unit vertical strips between the neutral axis and the compression flange. The final stress state solution was found by applying the total potential energy. A web-slenderness limit was developed in the form of:

$$\frac{D}{t_w} = \frac{36500}{\sqrt{F_y}}\left[1 - 8.6\frac{a}{R} + 34(\frac{a}{R})^2 \right] \tag{11}$$

where D is the web height, t_w is web thickness, a is panel length, and F_y is the yield stress in psi.

An extended version of the approach improved the cylindrical strips stiffness calculation by considering the whole web panel portion instead of discrete vertical strips (Culver et. al 1973). The following web-slenderness limit for curved girders was developed.

$$\frac{D}{t_w} = \frac{46000}{\sqrt{f_b}}\left[1 - 2.9\sqrt{\frac{a}{R}} + 2.2\frac{a}{R} \right] \tag{12}$$

where f_b is the flange bending stress in psi.

The only research related to the fatigue investigation of curved girders in the U.S. took place at Lehigh University. Two twin-girder assemblies, I-shaped and box, were analyzed only experimentally (Daniels et al. 1979a). The test assemblies were designed based on the early works on the straight girder fatigue requirements (Muller and Yen 1968). It was concluded that the Culver et al. equations were too conservative, and the following slenderness requirements were proposed:

$$\frac{D}{t_w} = \frac{36500}{\sqrt{F_y}}\left[1 - 4\frac{a}{R}\right] \le 192 \tag{13}$$

$$\frac{D}{t_w} = \frac{23000}{\sqrt{f_b}}\left[1 - 4\frac{a}{R}\right] \le 170 \tag{14}$$

Another research on fatigue testing of the horizontally curved girder was conducted in Japan (Nakai et al. 1990). A simplified method to approximate the maximum out-of-plane bending stress was developed and used for designing the scaled I-girder test setup. The curved web panel was idealized as a vertical strip with unit width, as illustrated in Fig. 3. The equivalent radial loading from in-plane bending was applied to the fixed-end strip beam model. The formula for maximum out-of-plane bending stress and displacements of the vertical strip was obtained by application of the beam theory:

$$\sigma_{b-\max} = \frac{1-\nu^2}{10}\sigma_0\left(\frac{a}{R}\right)\left(\frac{h_w}{a}\right)\left(\frac{h_w}{t_w}\right) \tag{15}$$

$$\delta_{w-\max} = \frac{\sqrt{5}(1-\nu^2)}{1250E}\left(\frac{a}{R}\right)\left(\frac{h_w}{a}\right)\left(\frac{h_w}{t_w}\right)^2 \sigma_{In-\max} h_w \tag{16}$$

where $\sigma_{b-\max}$ is the maximum out-of-plane bending stress, σ_0 is the maximum in-plane stress a is the panel width, ν is Poisson's ratio, t_w is web thickness, h_w is web height, $\delta_{w-\max}$ is the maximum lateral displacement, E is the elastic modulus, and R is the radius of curvature. Although equations 15 and 16 were verified by the finite displacement method, only a web panel aspect ratio of 0.7 was considered in both the experiments and analyses. In other words, equations 15 and 16 do not include the two-way action of the web panel. The applicability of equation 15 is discussed in the results section.

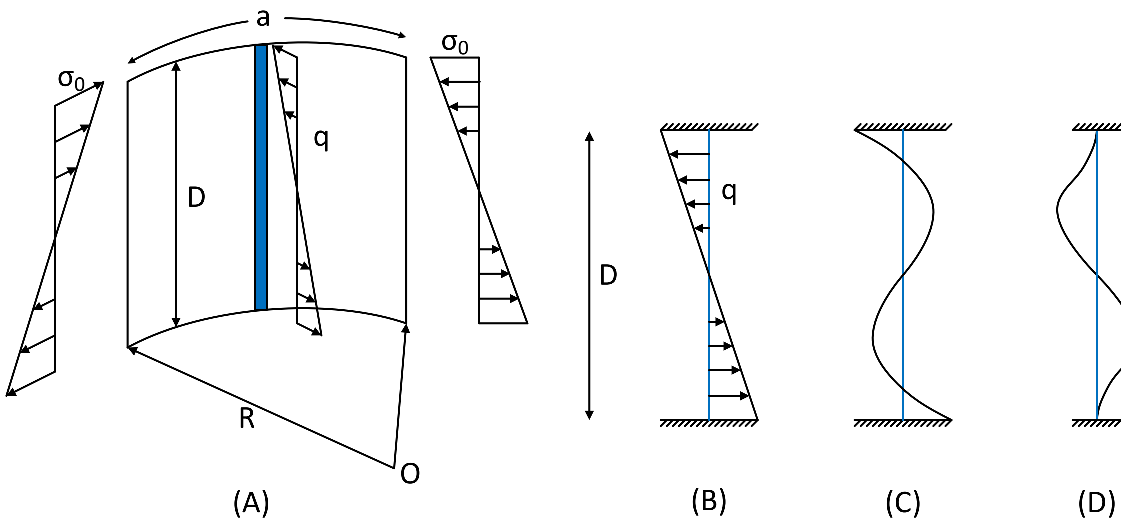

Figure 3. Analytical model of Nakai et al. 1990 A)loading condition B) lateral load on the vertical strip beam, C) stress distribution of the vertical strip, D) displacement distribution

Davidson et al. (1999a) developed a theoretical model referred to as the "lateral pressure analogy" to calculate the curved web panel bending stress and lateral displacement based on linear plate theory. The equivalent lateral load resulting from in-plane bending was applied to a flat panel with the same dimensions, as shown in Fig 4. The curved web panel behavior is similar to the flat plate under hydrostatic pressure with the G.D.E (Timoshenko and Woinowsky 1959):

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2 \partial y^2}+\frac{\partial^4 w}{\partial y^4}=\frac{q}{D}=\frac{q_0 x}{bD} \tag{17}$$

where w is the plate deflection function, b is the web height in compression, q_0 is the lateral pressure due to in-plane loading $q_0=\frac{\sigma_0 t}{R}$, σ_0 is the in-plane bending stress at the web-flange juncture.

The maximum web lateral displacement and bending stress were calculated based on the simply and fixed support condition for the top of the web (web-to-flange connection), respectively:

$$\delta_{\max}=\frac{\alpha h_c^4 \sigma_0 (1-\nu^2)}{Et^2 R} \tag{18}$$

$$M_{b\theta}=\frac{\lambda h_c^2 t \sigma_0}{R} \tag{19}$$

where α and λ are coefficients dependent on panel aspect ratio and the location of the displacement and bending, respectively, h_c is the web in compression. The coefficients were found using parametric FEM analyses.

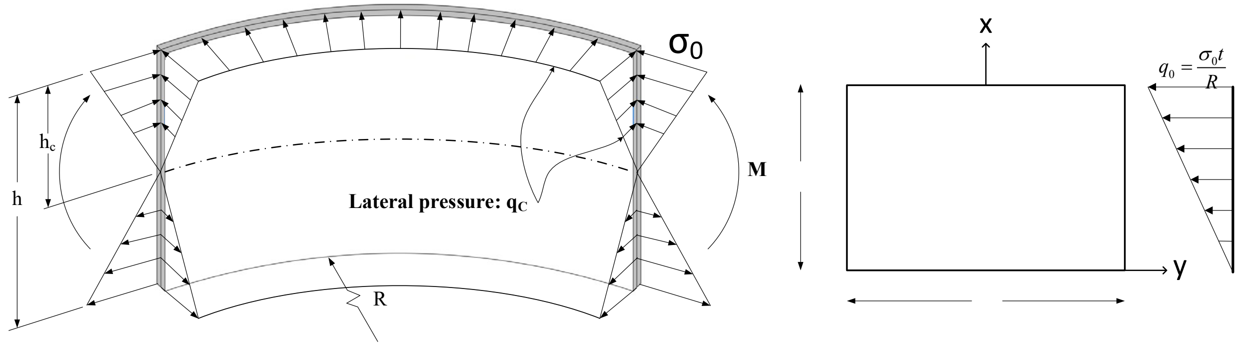

Figure 4. Lateral pressure analogy model of Davidson et al. (1999a)

4. Static loading vs. stress range

Calculation of the secondary bending stresses σ_b requires complex methods and cannot be defined by an ordinary equilibrium of applied loads. Hence, it is not practical to establish design guides directly based on limiting the secondary bending stress ranges to the web breathing fatigue strength, i.e., Eq. 20:

$$\Delta\sigma_b \leq \Delta\sigma_C \tag{20}$$

where $\Delta\sigma_b$ is the secondary bending stress range, $\Delta\sigma_C$ is the fatigue strength. A more appropriate method, developed by Maeda and Okura (1983), is to limit in-plane loads that prevent high out-of-plane bending stresses that can cause fatigue cracking. Fig. 5 represents a typical nonlinear relationship between the applied in-plane stresses and out-of-plane bending stresses for flat plates. The secondary bending stress increases more rapidly at higher levels of applied loads. Consequently, the smaller in-plane stress amplitude $\Delta\sigma_0$ of load condition 2 results in the same $\Delta\sigma_b$ of load condition 1.

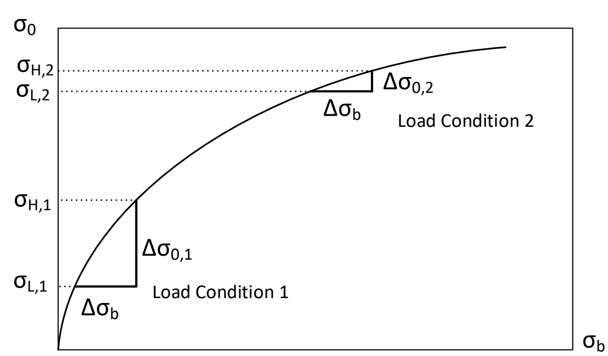

Figure 5. Normal stress σ_0 vs. secondary bending stress σ_b

The relationship between in-plane stress and stress range can be given by :

$$ST = \frac{\sigma_L}{\sigma_H} \tag{22}$$

$$\Delta\sigma_0 = \sigma_H(1 - ST) \tag{23}$$

where ST is in-plane stress ratio, $\Delta\sigma_0$ is the in-plane stress range, σ_L and σ_H is the minimum and maximum in-plane stress, respectively. Fig. 6 illustrates the Maeda and Okura (1983) approach for a stress ratio of 0.5. A slope triangle with a fixed base equal to $\Delta\sigma_C$ is moved along the curve until the equation 3 condition is met. The triangle with the solid lines shows the governing loading condition in which by applying the in-plane stress range $\Delta\sigma_0 = 0.5\sigma_H$, the amplitude of the secondary bending stress is equal to the fatigue strength $\Delta\sigma_c$. Consequently, the maximum in-plane bending stress can be defined in such a way that the secondary bending stress range does not lead to fatigue cracking.

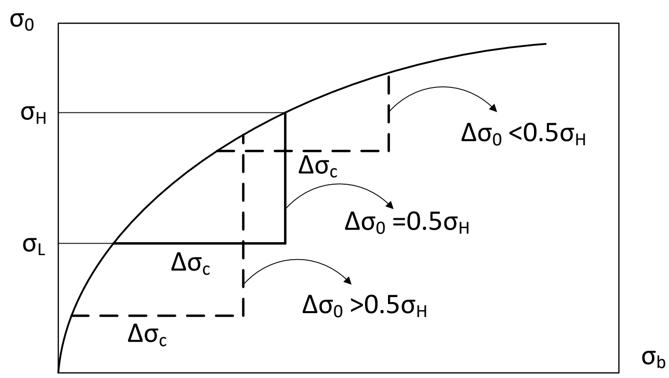

Figure 6. Maeda and Okura (1983) approach for finding the maximum σ_0

4. Result and Discussion

The current analytical investigation through the solution of the curved web panels GDE, Eq. 9 and 10, is under development. Preliminary FEM analysis is presented here and compared with the Nakai et al. (1990) analytical model, Eq. 15.

ABAQUS (2019) FEM package is used to simulate the curved web panel. 4-noded Shell elements, S4 elements, with full integration in the plane of the element, and 5 integration through the thickness was considered. The fine mesh is constructed by 100*100 elements through the depth and length of the web panel to properly capture the secondary bending stresses at the top and bottom of the web, as shown in Fig. 7. Bending stress in the plane of the web is modeled by the equivalent nodal forces in the tangential direction of the cylindrical coordinate system. The web panel dimensions and loading magnitudes are given in Table 1.

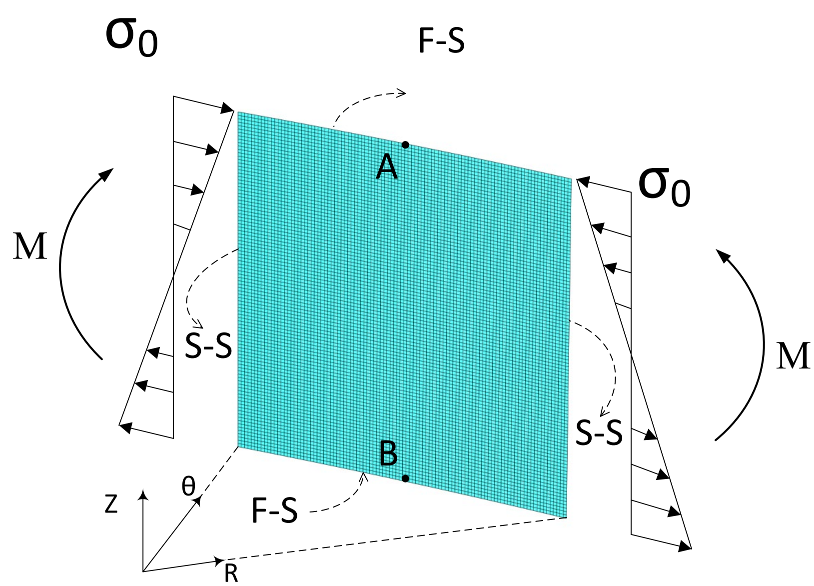

Figure 7. Load and boundary condition

Table 1. Web panel dimensions and loading condition

t_w (in)	D (in)	a (in)	R (ft)	$(\sigma_0)_{max}$ (ksi)	$(\sigma_0)_{min}$ (ksi)	$\Delta\sigma_0$ (ksi)	Elastic Modulus (ksi)	Slenderness ratio
0.668	100	100	100	36	24	12	29000	150

The maximum and minimum in-plane stresses are selected so that the resultant in-plane stress range, $\Delta\sigma_0$, is equal to the fatigue strength of crack type 2, $\Delta\sigma_c^{Type2}$. The secondary bending stress range, $\Delta\sigma_b$, is calculated based on the analytical equation (Nakai et al. 1990), linear, and nonlinear FEM simulation. The corresponding results are presented in Table 2.

Table 2. Secondary bending stress range comparison

	$\Delta\sigma_b$ (ksi)	$\Delta\sigma_b$ (ksi)
	Point A,B	Point C,D
Nakai Eq.	13.62	---
Linear FEM	13.68	---
Non-Linear FEM	15.13	24.32

The Nakai et al. equation accurately calculates the mid-panel linear response, and the nonlinear response can be approximated by applying proper amplification factors. The web normal stress in the transverse direction for the linear and nonlinear FEM analyses is shown in Fig. 8 for further explanation. The maximum web deformation and secondary bending stress occur at the mid-panel and points A/B, respectively. The deformation pattern is the same for the both minimum and maximum in-plane bending stresses, $(\sigma_0)_{\min}$ and $(\sigma_0)_{\max}$. However, the nonlinear displacement response experience a jump of buckling shape mode-1 to mode-3 in the compression region as the in-plane bending stress increases. Fuji and Ohmura (1985) first observed this phenomenon by solving the nonlinear G.D.E of curved web panels under bending. The maximum secondary bending stress range increases to almost double the linear analysis, 24.3 ksi compared to 13.6 ksi. This phenomenon that has significant effect on fatigue behavior of slender curved web panels is ignored by the simplified methods such as Nakai et al. (1990).

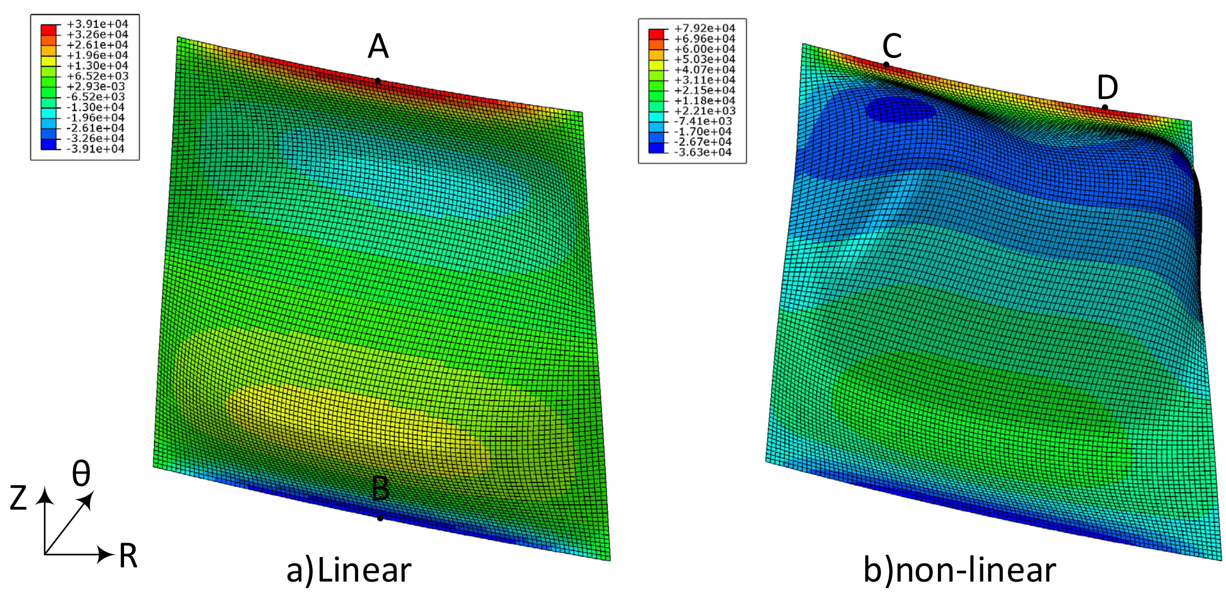

Figure 8. Web normal stress in transverse direction under maximum in-plane bending $(\sigma_0)_{\max}$, a) linear analysis b) geometric non-linear analysis (magnified deformation)

5. Summary and conclusion

Analytical methods for modeling the secondary bending stresses for flat web panels, web breathing, and stability methods of curved web panels were reviewed. The most accurate theoretical response of slender webs are defined by developing the solution of nonlinear governing differential equations based on plate and shell theory. However, the sophisticated mathematical solution procedures makes them less practical compared to simplified methods. It was shown that fatigue behavior of slender curved web panels requires nonlinear geometric models that take into account the 2-way action of shells and could not be fully recognized by the simplified methods based on beam theory.

Acknowledgements

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6. References

AASHTO, (1989) Standard Specifications for Highway Bridges, American Association of State Highway and Transportation Officials, 14th Edition.

ABAQUS, (2019), https://www.3ds.com. Dassault Systèmes Simulia Corp., Johnston, RI.

Aribert, J. M., Remadi, A., Raoul, J.(1996) “Parametric Study by FEM on Breathing of Steel Slender Webs” PD-Vol. 73, Third Biennial Joint Conference on Engineering Systems Design and Analysis, ESDA, Vol. 1, Montpellier, France.

Culver, C.G., Dym, C.L. and Brogan, D.K., (1972). Bending behavior of cylindrical web panels. Journal of the Structural Division, 98(10), pp.2291-2308.

Culver, C.G., Dym, C.L. and Uddin, T., (1973). Web slenderness requirements for curved girders. Journal of the Structural Division, 99(3), pp.417-430.

Dabrowski, R. (1968). “Gekrümte dünnwandige Träger: Theorie und Berechnung.” Springer-Verlag, Berlin.

Daniels, J. H., Zettlemoyer, N., Abraham, D., and Batcheler, R. P. (1979a). ‘‘Fatigue of curved steel bridge elements—analysis and design of plate girder and box girder test assemblies.’’ DOT-FH-11-8198.1, Lehigh University, Bethlehem, PA.

Davidson, James S., Scott R. Ballance, and Chai H. Yoo. (1999a). “Analytical Model of Curved I-Girder Web Panels Subjected to Bending.” Journal of Bridge Engineering 4 (3): 204–12.

Davidson, James S., Scott R. Ballance, and Chai H. Yoo. (1999b). “Finite Displacement Behavior of Curved I-Girder Webs Subjected to Bending.” Journal of Bridge Engineering 4 (3): 213–20.

Dubas P. (1992) “Zum Problem der Stegatmung bei dünnwandigen Blechträgern” Publikation Nr. 92-3, Baustatik und Stahlbau, Eidgenössische Technische Hochschule Zürich.

Fujii, K., and Ohmura, H.(1985) ” Nonlinear Behavior of Curved Girder-Web Considered Flange Rigidities”, Proceeding of JSCE Structural Engineering, No. 356, pp. 69-79.

Jalali, Mehran M., Justin D. Marshall, and James S. Davidson.(2020) "Buckling and distortion induced fatigue of curved steel plate girders with slender web." Proceedings of the Annual Stability Conference Structural Stability Research Council Atlanta, Georgia.

Kuhlmann, U., Günther, H.-P. (1999)“Zum Nachweis der Ermüdungsfestigkeit geschweißter Stahlträger mit schlanken Stegen” Research report, Institute for Construction and Design , University of Stuttgart.

Günther, H.-P., and U. Kuhlmann. (2004) “Numerical Studies on Web Breathing of Unstiffened and Stiffened Plate Girders.” Journal of Constructional Steel Research 60 (3–5): 549–59.

Linzell, D., Hall, D., & White, D. (2004). “Historical perspective on horizontally curved I girder bridge design in the United States”. Journal of Bridge Engineering, 9(3), 218-229.

Maeda, Y., Ishiwata, M., Kawai, Y.(1976) “Additional Tests for Type 2 Crack in Hybrid Girders” International Association for Bridge and Structural Engineering, Tenth Congress, Final Report, Toky, S. 337-340.

Maeda, Y., Okura, I. (1983) “Influence of Initial Deflection of Plate Girder Webs on Fatigue Crack Initiation” Engineering Structures, Vol. 5.

Maeda, Y., Okura, I. (1984) “Fatigue Strength of Plate Girder in Bending Considering Out-of-Plane Deformation of Web” Structural Eng./Earthquake Eng. Vol. 1, No. 2, S. 35-45.

Marguerre, K. (1938). “ Zur Theorie der gekrümmten Platte großer Formänderung.”
Proceedings Fifth International Congress on Applied Mechanics, Cambridge, Mass., S. 93.

Mueller, J.A. and Yen, B.T., (1966). Fatigue tests of large-size welded plate girders, June 1966, Publication No. 311 (66-19).

Mueller, J. A. and Yen, B. T., (1968). "Girder web boundary stresses and fatigue, WRC Bulletin No. 127, Reprint No. 68-2" Fritz Laboratory Reports. Paper 260.

Nakai, Hiroshi, Toshiyuki Kitada, Hiroshi Ishizaki, and Katsuyoshi Akehashi. (1990). “An Experimental Study on Fatigue Strength of Web Plates for Horizontally Curved Plate Girders.” Doboku Gakkai Ronbunshu, no. 416.

Patterson, P. J., Corrado, J. A., Huang, J. S., & Yen, B. T. (1970). Fatigue and static tests of two welded plate girders. Welding Research Council Bulletin, 155, 1-18.

Remadi, A., Aribert, J. M., Raoul, J.(1995) “Secondary Bending Stresses due to out-of-Plane Displacement of Slender Web Panels Subject to a Bending Moment” International Colloquium, Stability of Steel Structures, Budapest, Hungary, S. 277-284.

Roberts, T. M, and A. W Davies. (2002). “Fatigue Induced by Plate Breathing.” Journal of Constructional Steel Research 58 (12): 1495–1508.

Spiegelhalder U. (2000)" Zur materialermudung infolge stegatmung". Doctoral Thesis, Institut fur Konstruktion und Entwurf I der Iniversitat Stuttgart, Stuttgart, Germany.
Timoshenko, S., and Woinowsky-Krieger, S. (1959). Theory of plates and shells. McGraw-Hill, New York.125–33.
Wachowiak, T. (1967). ''Die Berechnung gekruemmter Stege duennwandigen traegern auf Grund der Shalentheorie,'' PhD thesis, Politechnika Gdanska, Gdansk, Poland.
Washizu, K. (1975). "Variational methods in elasticity and plasticity", 2nd Ed., Pergamon Press, Tarrytown, N.Y.