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Critical moment of doubly-symmetric beams with prebuckling deflection: the effect of intermediate supports

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Abstract
In the reported research the effect of prebuckling in-plane deflections on the critical moment to lateral-torsional buckling of beams is investigated. The paper focuses on the influence of support conditions, particularly that of intermediate discrete lateral supports. Parametric numerical and analytical studies are presented: critical moment values have been calculated with and without considering the effect of prebuckling deflection, using derived analytical formula, and finite element method. One of the most important conclusions is that – unlike suggested by earlier literature, – the prebuckling deflections are not always positive, but for certain supports can decrease the critical moment. Another important conclusion is that the calculated efficiency of an intermediate lateral support is greatly dependent on whether the prebuckling deflection is considered or not.

1. Introduction

Lateral-torsional buckling (LTB) can be the governing behavior/failure mode for a laterally unsupported beam. Linear buckling analysis (LBA), leading to the critical load(s) and buckling shape(s), alone is typically not enough to characterize the load-bearing capacity, still, LBA is a useful tool to understand the behavior. In addition, the critical load (e.g., critical moment) is widely used in design calculations.

In classic LBA, the equations – either differential equations or energy equations – to be solved are written on the original, undeformed geometry. However, the primary, first-order deformations might influence the solution. As the load is increased on the structure, the primary deflections increase, and by the time when buckling occurs, the structure is already deflected. This deflected shape is referred to here as *prebuckling deflection*. It is to understand that the prebuckling deflection is not an imperfection, since it is due to the loading, and it exists even if the original structure is perfect. If the prebuckling deflection is considered in the LBA, the associated critical load is different from the one without considering it. It is reasonable to assume that prebuckling deflection is never zero, however, whether it has important or negligible effect on the buckling, is dependent on the structure.

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In the case of beams subjected to LTB, the effect of the prebuckling deflection was included even in the very first analytical solution for the LTB problem by Michell (1899). Later, the solution without the prebuckling deflection effect became widely known from the work of Timoshenko (1906) who clarified the importance of warping, and published the well-known classic formula for the critical moment for LTB. The influence of prebuckling displacement, then, was discussed in a relatively small number of papers; a review is provided in Abureden and Adany (2025a).

There seems to be a consensus in earlier literature that the prebuckling deformations increase the critical moment, and that the increase is dominantly determined by the lateral rigidity of the beam. However, there are some discrepancies, too, both in the proposed analytical expressions and in the numerical results. Moreover, these previous researches mostly focused on the simplest case of LTB, namely: simply supported single-span beams, subjected to uniform bending, and with doubly symmetric I-shaped cross-sections. Other cases were hardly investigated. In the research reported here the aim is to expand our understanding on how the prebuckling deflection influences the LTB phenomenon.

This paper focuses on the effect of supports, including intermediate discrete lateral supports. In Section 2 the completed calculations are briefly summarized. The results of the numerical studies are summarized in Section 3: critical moment values with and without considering the effect of prebuckling deflection are calculated, using analytical formulae and beam finite element method. Finally, in Section 4 the main conclusions are drawn.

2. Scope

2.1 Beams considered

In this study simple beams are analyzed with doubly symmetric cross-sections, subjected to uniform moment distribution along the length, see Fig. 1. Various end supports and intermediate lateral supports are considered. Regarding the end supports, both are pinned in the primary plane, i.e., both beam ends can freely rotate about the *x*-axis. The restraints against the *y*-axis rotation and warping are either ‘pinned’ or ‘fixed’. If the end of the beam is free to rotate in the lateral direction (i.e., pinned around the *y*-axis), it is denoted as ‘Pr’, while if it is fixed against lateral rotation, it is denoted as ‘Fr’. Similarly, whether the warping can freely occur or prevented, it is ‘Pw’ or ‘Fw’. Accordingly, for example, PrPw-PrPw identifies the basic case of lateral-torsional buckling, when forked supports are applied. Or PrFw-PrFw is the case when the lateral rotations are allowed but the warping is restrained at both ends.

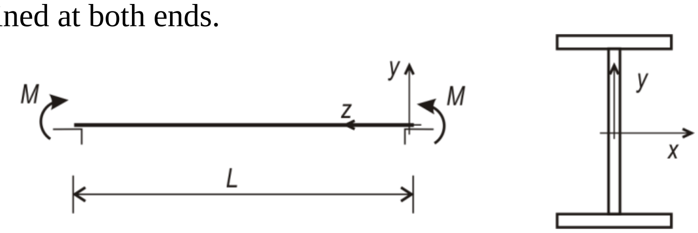

Figure 1: Intermediate discrete lateral supports considered

The middle cross-section of the beam is laterally supported, in one of the configurations as follows: Top flange Laterally Supported (TLS), Centroid Laterally Supported (CLS), Bottom flange Laterally Supported (BLS), and All the cross-section Laterally Supported (ALS). Fig. 2 illustrates these cases. As a reference, the No Lateral Supports (NLS) case is considered, too. It is to emphasize that ‘lateral support’ means that the lateral translation is prevented at one (or multiple) cross-sections points, see Fig. 2. As a result, in the case of CLS the twisting rotation of the cross-section can freely occur; in the case of TLS and BLS the twisting rotation of the middle cross-section is not prevented but is linked to the lateral translations; while in the case of ALS the twisting rotation of the middle cross-section is fully prevented.

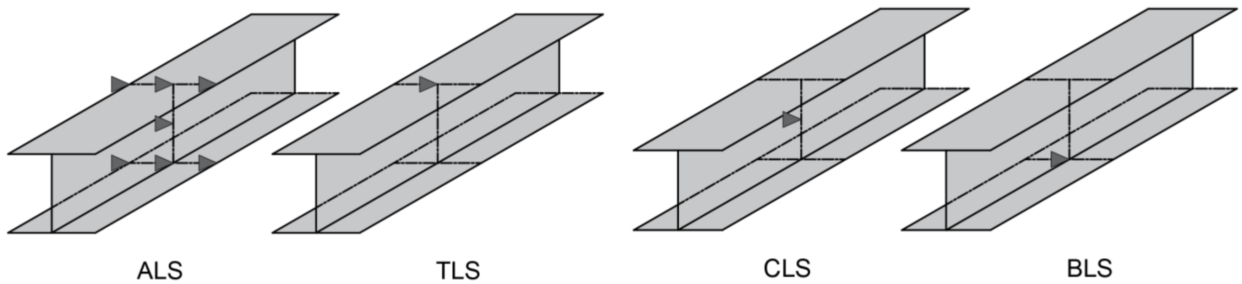

Figure 2: Intermediate discrete lateral supports considered

Previous studies showed that the prebuckling effect is majorly dependent on the ratio of the minor-axis and major-axis flexural stiffnesses (i.e., the EI_y/EI_x=I_y/I_x ratio). Accordingly, several cross sections are considered so that the flanges remain the same (200×20 mm), the web thicknesses remains the same (12 mm), but the depth of the cross-section varies (between 150 and 500 mm). This results in I_y/I_x ratios between 0.05 and 0.75. The L beam length varies, too (between 2.5 and 50 m), i.e., the length range is extremely wide. Isotropic steel is considered, with a Young’s modulus equal to 210000 MPa, and Poisson’s ratio equal to 0.3.

2.1 Methods applied

To calculate the critical moment values with and without the prebuckling effect, analytical and numerical methods have been employed. New critical moment formulae have been derived using the energy method, following the principles proposed in Pi and Trahair (1992).

For numerical analyses, the finite element method (FEM) has been employed, using beam finite elements. (Note, shell elements have also been used, but these results are not discussed in this paper.) For the FEM analyses the Ansys APDL (Ansys, 2020) has been used which offers the BEAM188 element, a 2-node element with 7 degrees of freedom at each node (three translations, three rotations, and the warping degree of freedom). When the prebuckling deflections are not considered, a classic linear buckling analysis (LBA) can be completed, directly included in most FEM software implementations. However, to obtain the critical moment with the prebuckling effect needs an iterative LBA procedure, as follows.

Step 1: Classic LBA, to calculate M_{cr} (which, in this step, will be equal to M_{cr0}).

Step 2: Linear static analysis to get the deflected shape (i.e., the prebuckling shape), using the M_{cr} as load from the previous Step.

Step 3: LBA on the deflected shape, using the deflected shape from the previous Step, from which a new value for M_{cr} is obtained.

Then Steps 2 and 3 can be repeated till convergence.

3. Results: the effect of prebuckling deflections

In earlier papers multiple formulae were proposed to calculate the effect of prebuckling deflection on the critical moment. However, by introducing appropriate simplifications, as detailed in in Abureden and Adany (2025a), most of the previously proposed formulae take the following format.

$$M_{cr} = M_{cr0} \Big/ \sqrt{\left(1 - \frac{I_y}{I_x}\right)} \tag{1}$$

where M_{cr0} and M_{cr} is the critical moment without and with the effect of prebuckling deformations considered, I_x and I_y are the second moments of area for the *x* (major) and *y* (minor) axes, respectively. This formula suggests that (i) M_{cr} is larger than M_{cr0}, i.e., the prebuckling deflection increases the critical moment, and (ii) M_{cr} monotonously increases as I_y/I_x increases. The (relative) moment increase, therefore, can conveniently be expressed as:

$$\frac{M_{cr} - M_{cr0}}{M_{cr0}} = 1 \Big/ \sqrt{\left(1 - \frac{I_y}{I_x}\right)} - 1 \tag{2}$$

The above formula predicts a significant moment increase if the inertia ratio is large, e.g., the increase is 100% when $I_y/I_x = 0.75$. Moreover, the critical moment tends to infinity as I_y approaches I_x, i.e., minor-axis LTB is nonexistent. It is also to observe that the moment increase is independent of the beam length.

The above formula can be justified for simply supported beams with open cross-sections, but not for other cases, see e.g., Abureden and Adany (2025b). The moment increase is greatly influenced (i) by the supports and (ii) by the length. This is illustrated in Fig. 3, where the results of beams with simple (forked) end supports are presented, considering various intermediate discrete lateral supports.

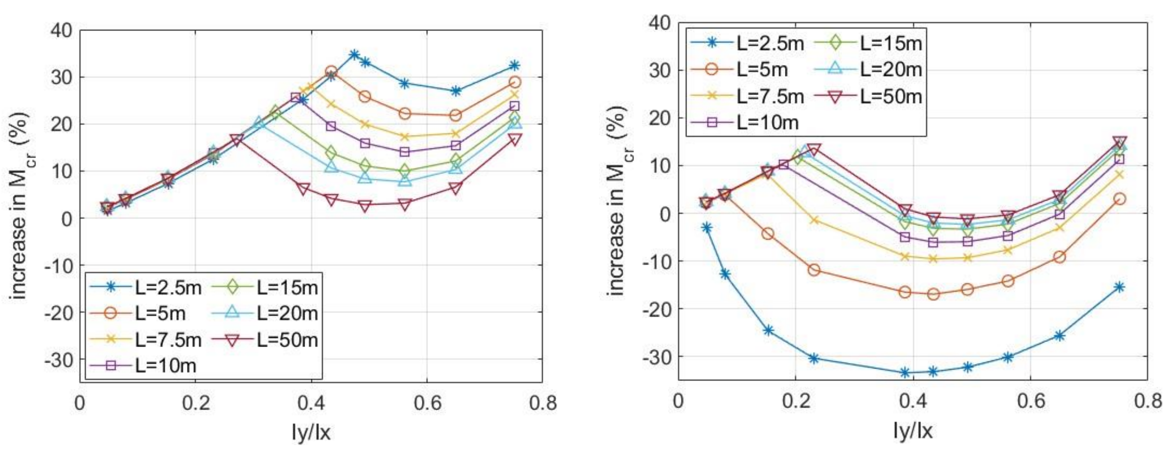

Figure 3: The effect of beam length on the moment increase: TLS (left) and CLS (right)

There are important observations as follows. (i) Typically, the introduction of some discrete lateral support reduces the moment increase. The reduction can be drastic. For example, when $I_y/I_x = 0.75$, the moment increase without any intermediate support is 100%, but it can be a negative increase in the presence of a centrally place lateral support. (ii) When an intermediate support is present, the beam length has significant effect. (iii) The tendency of the moment increase curves is far from being monotonously increasing. This can be explained by the buckling shapes. The buckling shape can be symmetric or point-symmetric; the various shapes are differently affected by the prebuckling deflections, and sometimes there is mode switch when the prebuckling deflection is considered. (iv) The position of the discrete lateral support (within the cross-section) has significant effect.

The results are even more scattered if the end support is different from simple (forked) one. This is demonstrated by Fig. 4, where the moment increase values for 30-m-long beams are plotted. The important observations are as follows. (i) In general, the warping fixity has small effect. This observation is in full accordance with the results presented in Abureden and Adany (2025b) for beams without intermediate lateral supports. (ii) In the case of ALS, the warping fixity has marginal influence, while the rotation fixity (about the minor axis) reduces the moment increase. (iii) If the rotation is restrained at one end only, the results are in between the Pr-Pr and Fr-Fr cases. (iv) In the case of TLS/CLS/BLS, the curve shapes are not significantly affected by the end supports, but the curves with end fixity run below the PrPw-PrPw curves, thus, the end fixity reduces the moment increase. Since, in general, both the end fixity and the intermediate lateral support decreases M_{cr}, it finally can be significantly smaller than M_{cr0}, the maximum decrease being 25-35%, depending on the lateral support position. (It is to note that for other beam lengths even larger moment degradation can be found.)

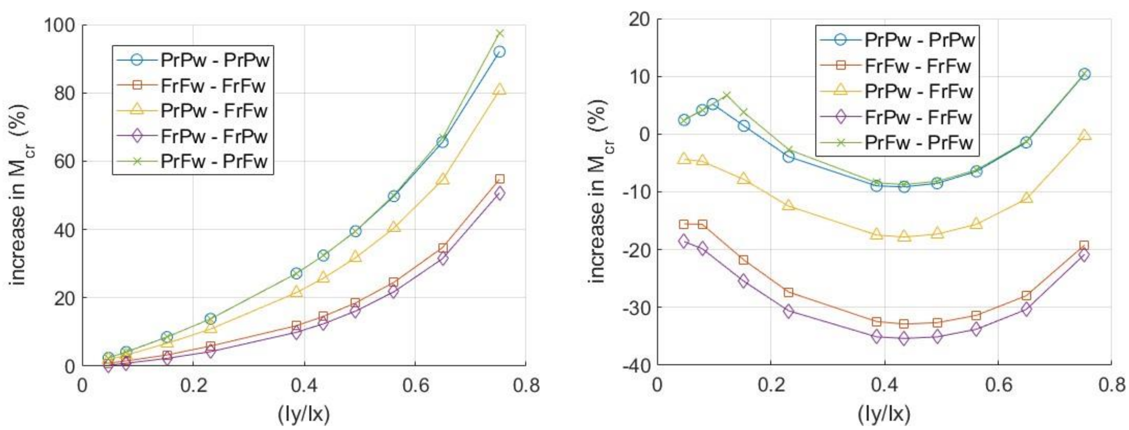

Figure 4: The effect of end supports on the moment increase: ALS (left) and BLS (right)

3. Results: the efficiency of the discrete intermediate lateral supports

Intermediate lateral supports are widely used in structural engineering constructions. They are known to enhance the LTB behavior. It is also known that the position and type of the intermediate support influences its efficiency. Based on the here-presented results, however, a further aspect is revealed: the efficiency highly depends whether the prebuckling deflections are disregarded or considered. To illustrate this, Figs. 5-6 show the efficiency for two sample cases. The efficiency is expressed by the $M_{cr0}/M_{cr0(NLS)}$ or $M_{cr}/M_{cr(NLS)}$ ratio, where M_{cr0} and M_{cr} are the critical moments with the lateral supports, while $M_{cr0(NLS)}$ and $M_{cr(NLS)}$ are the critical moments without any lateral intermediate support.

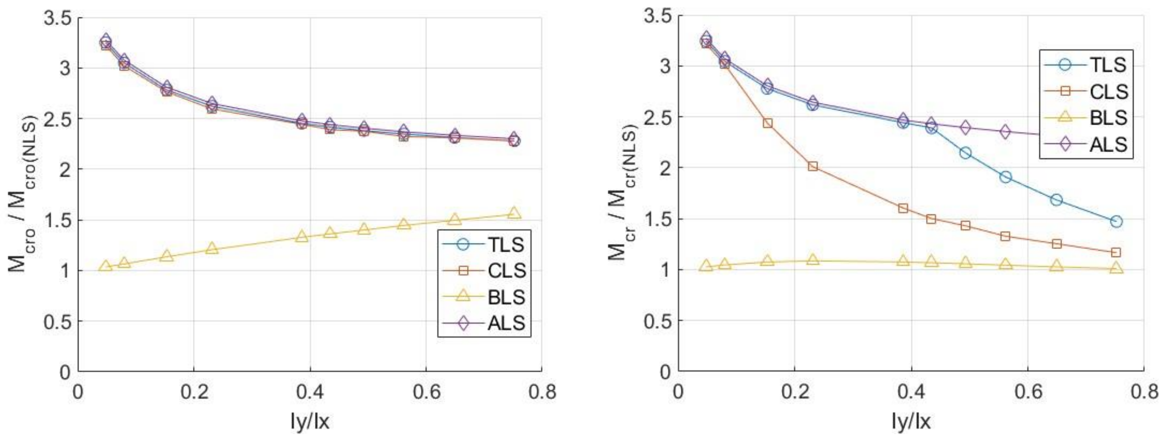

Figure 5: Ratio of critical moments with and without intermediate support for L = 5 m, PrPw-PrPw: M_{cr0} (left) and M_{cr} (right)

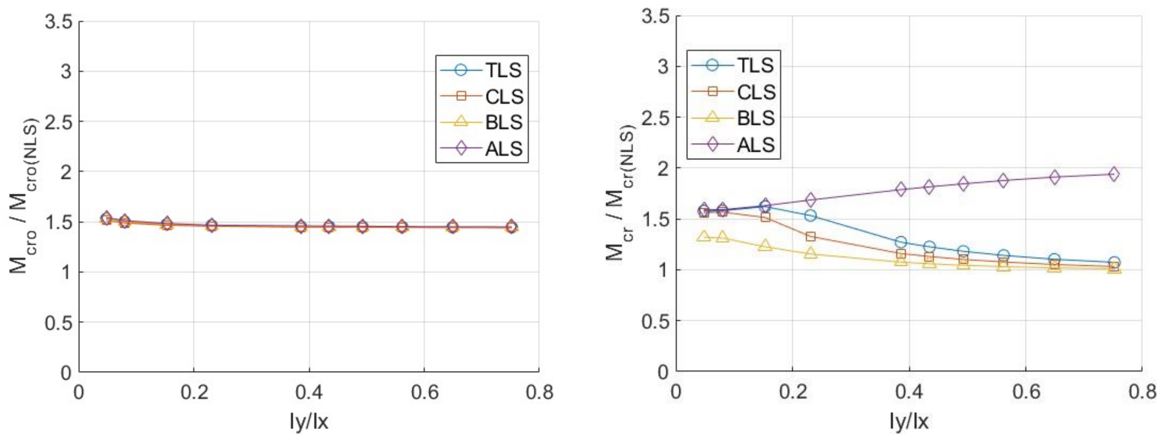

Figure 6: Ratio of critical moments with and without intermediate support for L = 30 m, FrFw-FrFw: M_{cr0} (left) and M_{cr} (right)

The most important observation is the striking difference between the left (without considering the prebuckling effect) and right (with considering the prebuckling effect) plots. In general, as expected, ALS is the most efficient and BLS is the least efficient, but the actual effect of the lateral support is strongly dependent on the problem parameters: the efficiency ratio can be as high as 3.3, but in many cases, it is hardly greater than 1. Moreover, there is no simple rule to know whether M_{cr0} or M_{cr} benefits more from the presence of some intermediate lateral support. For example, in the FrFw-FrFw case shown in Fig. 6, for large and moderately large I_y/I_x ratios, the efficiency of BLS is 50% and 0% without and with considering the prebuckling effect, respectively; however, the efficiency of ALS is 50% and 80% without and with considering the prebuckling effect, respectively. (Hence, the classic analysis predicts the same 50% moment increase for both BLS and ALS, but a more realistic prediction is 0% for BLS and 80% for ALS.)

4. Conclusions

In the reported research the LTB behavior is investigated using numerous assumptions and simplifications, such as: the material is perfectly elastic and homogeneous, the effect of imperfections is disregarded, only classic beam-type displacements are assumed (i.e., the effect of shear of localized plate bending deformations are disregarded), etc. Real beams are affected by all these factors. Still, it is a common notion that the results from an idealized elastic analysis are representative if the beam cross-section is compact and the beam is slender, i.e., it is expected that the critical moment values approximate the real LTB capacity in the case of locally compact but globally slender beams. However, the presented results illustrate that the critical moment value can strongly be affected by the prebuckling deflections. Depending on the structural configuration (most importantly: on the end and intermediate supports), the ratio of the critical moments with and without considering the prebuckling effect is in the range of 0.6-2.0 for practical doubly symmetric beams. Similar conclusions can be reached observing the efficiency of the intermediate lateral supports. The calculated efficiency of the lateral support is sometimes significantly different depending on whether the prebuckling effect is considered or not. Though further investigations are necessary, the obtained results suggest that elastic linear buckling analysis results are not always good in predicting the lateral-torsional buckling behavior even if the behavior is primarily elastic.

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