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Why do cantilever beams buckle the wrong way?

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Abstract
The displacement of cantilever beams as they undergo lateral torsional buckling is generally understood to be the opposite of the displacement of members which have a support at each end of the beam. In the case of the simply supported beam with a constant moment, the beam will buckle to a shape where the compression flange moves further from the original centre line of the beam than the tension flange does. For cantilever beams loaded with a point load at the tip, it is the tension flange which moves further during buckling. However, it may be difficult for a design engineer to see why this difference occurs, or why it matters. There is no easily accessible reference which explains this difference in behaviour. This paper will illustrate why there is a difference between the two loading situations. The development of the discussion will reference the derivation of the classic lateral torsional buckling expression, the assumptions attendant in that derivation, the behaviour of the compression and tension flanges within the Brazier Effect, and how those conditions change when the boundary conditions of the cantilever beam apply. The cantilever buckling behaviour is contrasted with the simply supported case. Some discussion of current bracing recommendations is presented.

1. Introduction
The following discussion presumes a wide flange “I” shaped beam, doubly symmetric, with shear centre and centroid coincident. The term flange can be extended to include the portion of the beam's cross section to the centroid for the stress regime (tension or compression) under consideration.

The impetus for this paper came from a discussion regarding the physical behaviour that leads to lateral torsional buckling of flexural members.

1.1 Brazier Effect
To illustrate the behaviour of isolated flanges in flexure, we'll first look at a different mode of instability. The concept of a through thickness force caused by simple bending is developed by Brazier (1927) in his work on ovalisation. Ovalisation is a localised cross section instability created by the development of through thickness compression forces that “squeeze” the member as it is bent, and at a critical loading will cause a crinkling of the member. While this phenomenon is not

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directly relevant to lateral-torsional buckling, there is a key concept in its derivation that will assist with the development shown later on with the direction of beam flange displacements during lateral torsional buckling and so is included here for illustration.

When a Bernoulli-Euler beam is bent by a constant moment, the flange forces will be constant along the length of the beam. The flanges will deflect in the direction of flexure, inclined to the original direction of the flange. With a constant moment, the curvature of the beam is also constant and the slope of a beam relative to the ends of a section will vary as the section's length.

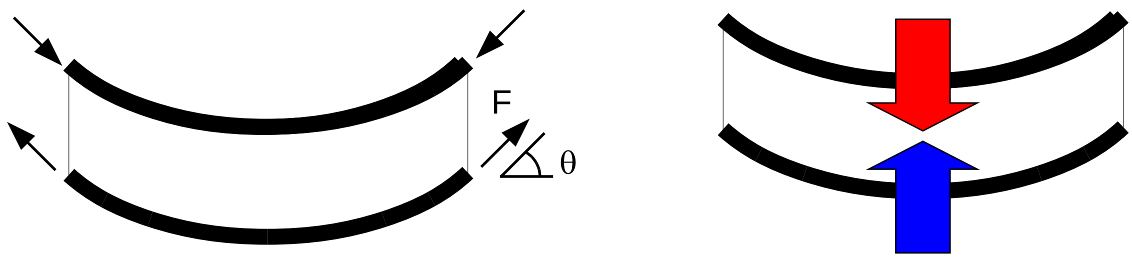


Figure 1: Flange forces in the Brazier Effect

The flange in compression will experience a lateral force toward the convex face, and the flange in tension will experience a force pushing to the concave face. The combination of these forces (and related displacements) will cause a compression force to be experienced by the beam through its depth. This through depth compression will be referred to as the “Brazier Effect.” The force directions noted above will be used to simplify the description of the flange behaviour under constant moments for beams experiencing lateral-torsional buckling.

1.2 Lateral Torsional Buckling Theory

Lateral torsional buckling was first described in the independent works of Michell and Prandtl in 1899 and refined by work by Timoshenko in 1905. (Timoshenko 1953)

The modern derivation of the lateral–torsional buckling moment is presented in a number of references, but as a generally available work, Timoshenko and Gere (1961) is a good exemplar. The general process is to take the deformed shape of a buckled beam and measure the components of the applied moments in the displaced local coordinate system. When the rotation of the beam is considered, there is a component of the applied moment causing lateral bending; and when the lateral bending of the beam is considered, there is a component of the applied moment that causes a torque about the long axis of the beam. In both deformed conditions, the major component of the applied moment is still producing bending about the beam's original axis of bending. Using small displacement theory, the applied moment bending will still be considered to be fully applied to the major axis.

Considering the deformations of the beam as it bends perpendicularly to the applied moment, a portion of the applied moment M_0 causes a torque about the long axis with a magnitude of $M_0 v'$, where v' is the slope of the beam's lateral displacement. Considering twist of the beam, the component of the applied moment causing lateral bending is $M_0\theta$ where θ is the rotation of the beam. Both v' and θ will vary along the length of the beam. The constitutive expressions for the response of the beam to these moment components give the simultaneous expressions

$M_0 v' = GJ\theta' - E\ C_w \theta'''$ for the lateral bending leading to a twist, and $M_0 \theta = -E\ I_y \mathrm{v}''$ for the twist leading to lateral bending. Combining these to get a single expression in one deformation variable is complicated by the introduction of $M_0 v'$ or shear times lateral slope. This complication can be removed by establishing M_0 as a constant value, with no shear present. The common expression of lateral torsional buckling is derived with that loading assumption.

$$M_{cr} = \sqrt{\frac{\pi^2 E I_y}{L^2}} \sqrt{GJ + \frac{\pi^2 E C_w}{L^2}} \tag{1}$$

For the derivation, the beam is assumed to be in bending without shear. The flanges have constant forces along their entire length. This is similar to the condition that is experienced on the through thickness compression of the beam described for ovalisation. The same criterion applies and the flange in compression will experience a lateral force directed toward the beam face bent in a convex shape, and the flange in tension will experience a force pushing toward the concave side of that flange.

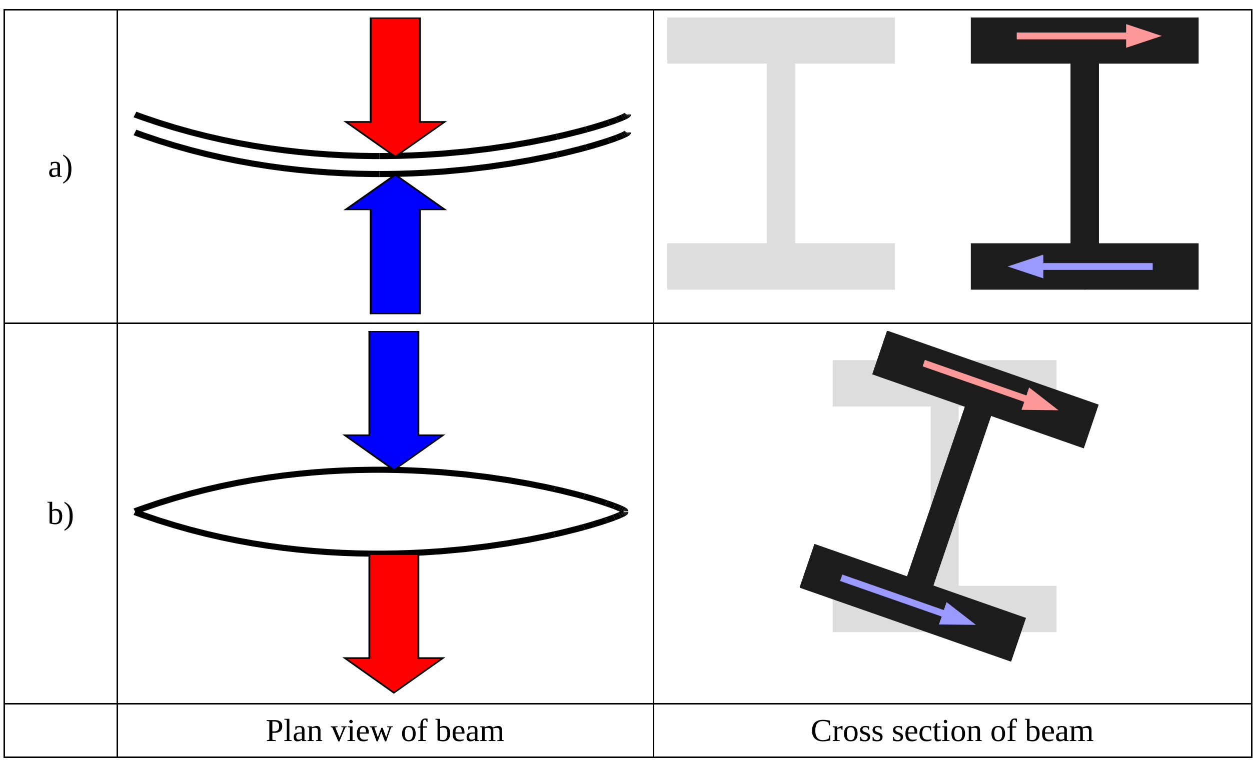


Figure 2: Flange forces in response to lateral torsional displacements

Deflection of the cross section due to the relative displacement of the flanges can be illustrated by the simplified schematic diagrams shown in Fig. 2. Here, the lateral displacement (a) and the rotational twist (b) are displayed separately, with the general forces in red for the compression flange and blue for the tension flange. The tension flange has two tendencies, one to move away from the side that the compression flange moves to (in rotation, as a response to lateral bending), and the other to move toward the side that the compression flange moves to (in flexure, as a response to the rotation of the beam). This could be thought of as three influences pushing the beam to the same direction as the compression flange moves, and one in the opposing direction. The compression flange simply moves in that one direction, the same direction as the imposed displacement, thus the compression flange has the larger lateral motion.

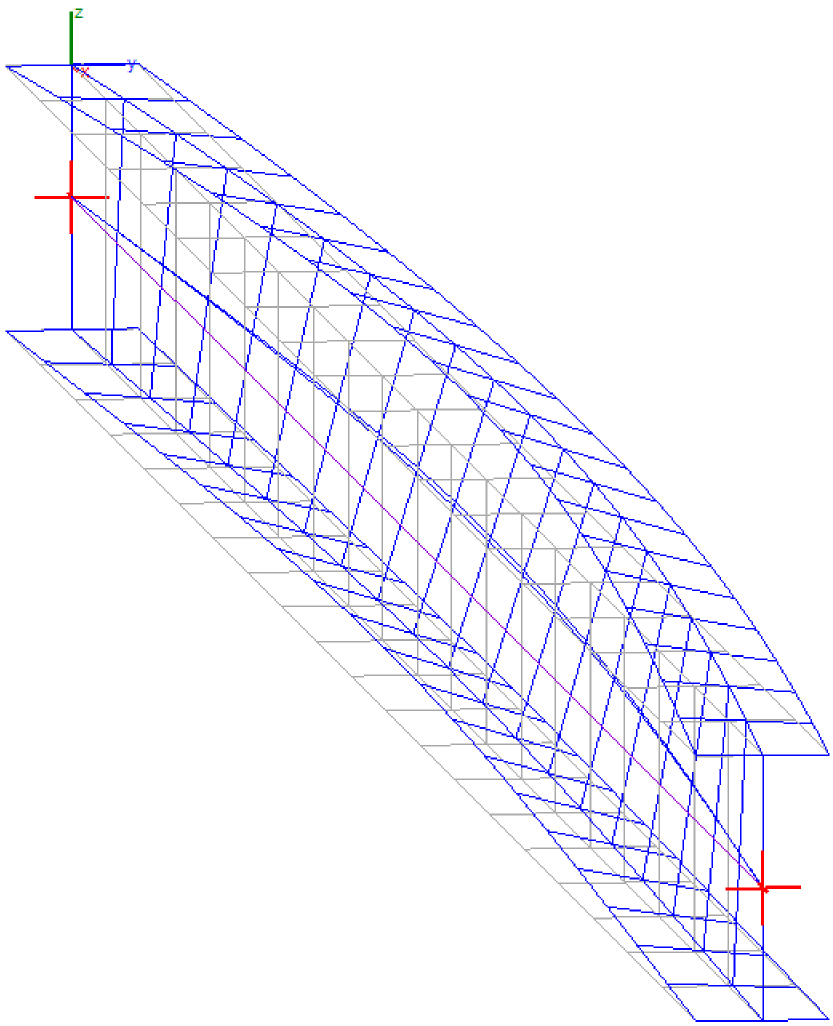


Figure 3: Buckled shape for simply supported beam, constant moment, compression in top (LTBeamN)

2. Cantilever beam deflections

In the case of a cantilever, the simplification of a constant moment no longer applies on a practical level. "The uniform bending condition of simply supported beams used for design does not occur for practical cantilevers, for which the most critical loading is that of concentrated load at the free end." (Trahair, 2020) the end moment cannot be applied easily and would not occur under any usual load cases. There is no continuous force in either flange, and the longitudinal forces are zero at the cantilever tip. The flange curve response described for ovalisation is not valid for these end conditions, and determining how the beam cross-section behaves requires more investigation.

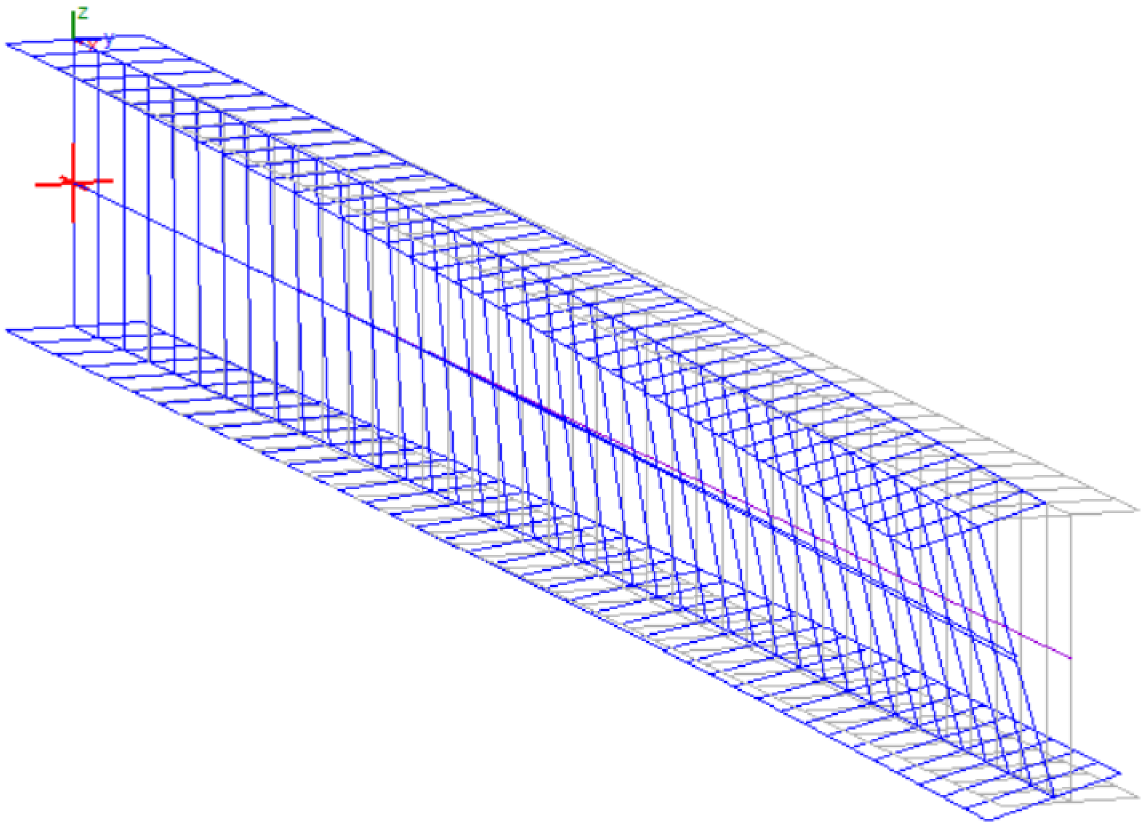


Figure 4: Buckled shape for cantilever with tip point load at shear centre, compression in bottom (LTBeamN)

At the tip of the cantilever, there is no force acting on either flange. Without compressive forces acting on the end of the compression flange, it has no destabilising tendencies. When that flange moves laterally, there will be a restoring force acting on the cantilever near the tip, “pushing” it back into its original alignment. On the tension flange, there is no resistance to the “pull” of the flange back toward the cantilever root, so the tension side will deflect further in the direction of any lateral displacement. In each case, the lateral component of the force has the magnitude of the flange force at that location times the sine of the angle of deviation. That angle is the slope of the bending curve about the weak axis. (See Fig. 5.)

This is opposite to the displacement directions for the members that have a moment present for the entire member (i.e. members that have two supports) where the compression flange will have a compressive force at the ends forcing that flange to bend laterally away from the concave side of the flexural curve, and the tension flange has a tension force acting at both ends pulling it to move away from the convex side of the flexural curve.

This combination of twist and bending actions produces a set of three influences moving the beam flanges in the same direction as the tension flange’s motion and one influence in the other direction, producing a net displacement in the same direction as the tension flange. Or in simpler terms, the tension flange is pushed out further than the compression flange. (See Fig. 6.)

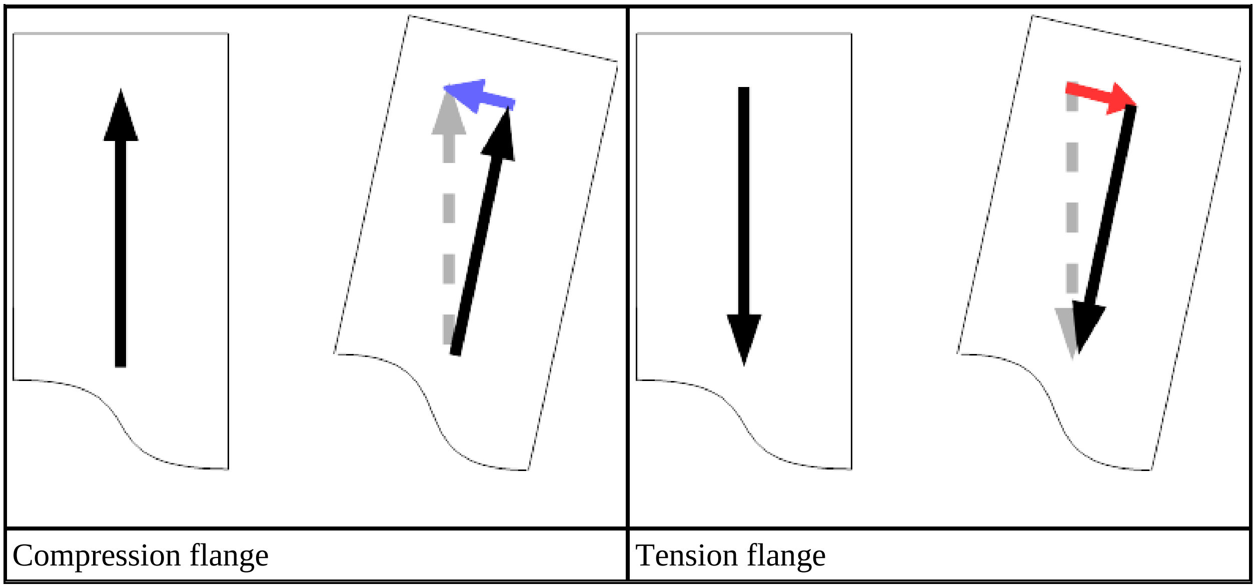


Figure 5: Cantilever tip forces before and after deflection

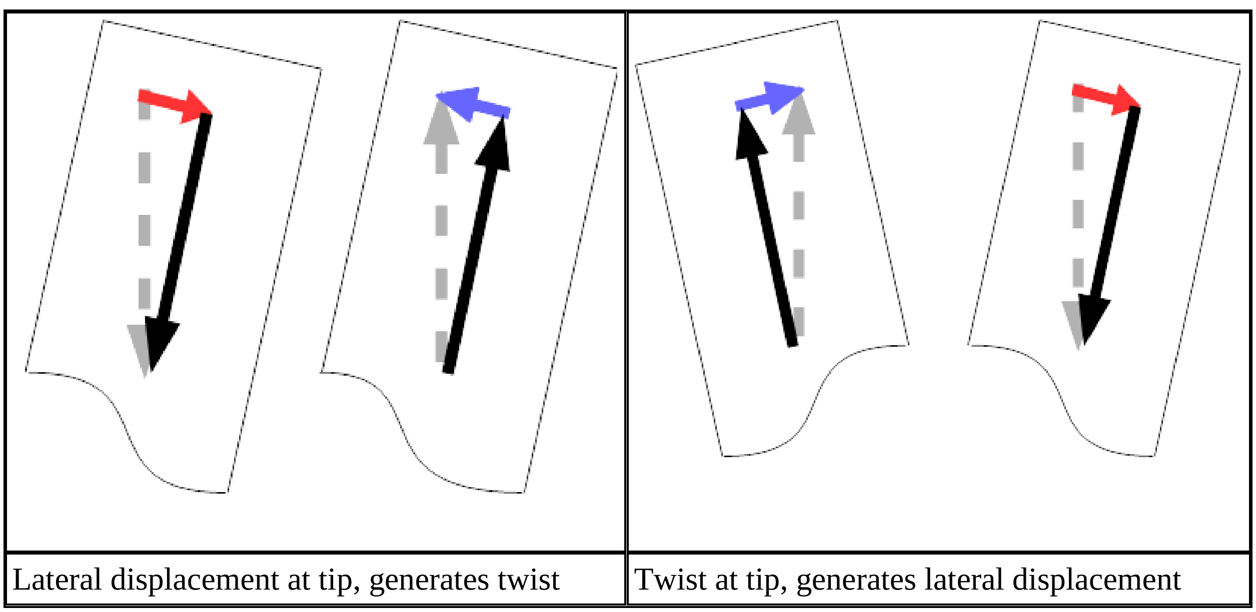


Figure 6: Cantilever tip deflections and beam responses

3. Alternative explanations

There may be other possible explanations for the cantilever behaviour, and these shall be explored briefly.

3.1 Load placement / height of loading

As the load on the cantilever will primarily be from a vertical (gravity) load applied at the tip, there may be an extra torque imparted, augmenting the rotational displacement that is pushing the tension flange further. This could be the case with "top flange" loading, where, as in beams supported at both ends, the extra torque is destabilising. With beams loaded at the shear centre, as was the case with the analyses shown here, there should be no extra localised torque applied.

3.2 Uniform moment on cantilever.

Because we've looked so intensely at the uniform moment earlier with beams and buckling tendencies, we should examine if that is still relevant. While it would be difficult to apply an end moment to the cantilever in practice, it is extremely easy to do so in simulation. If we apply a uniform moment that produces negative moment (tension in the top fibre) we do get a case where the bottom (compression) flange is the critical flange and moves further from the original position. (See Fig. 7 for an example of this displacement.) The theory derived for simply supported conditions and loads is still applicable in the abstract, but not in the case of reasonable load cases for cantilevers.

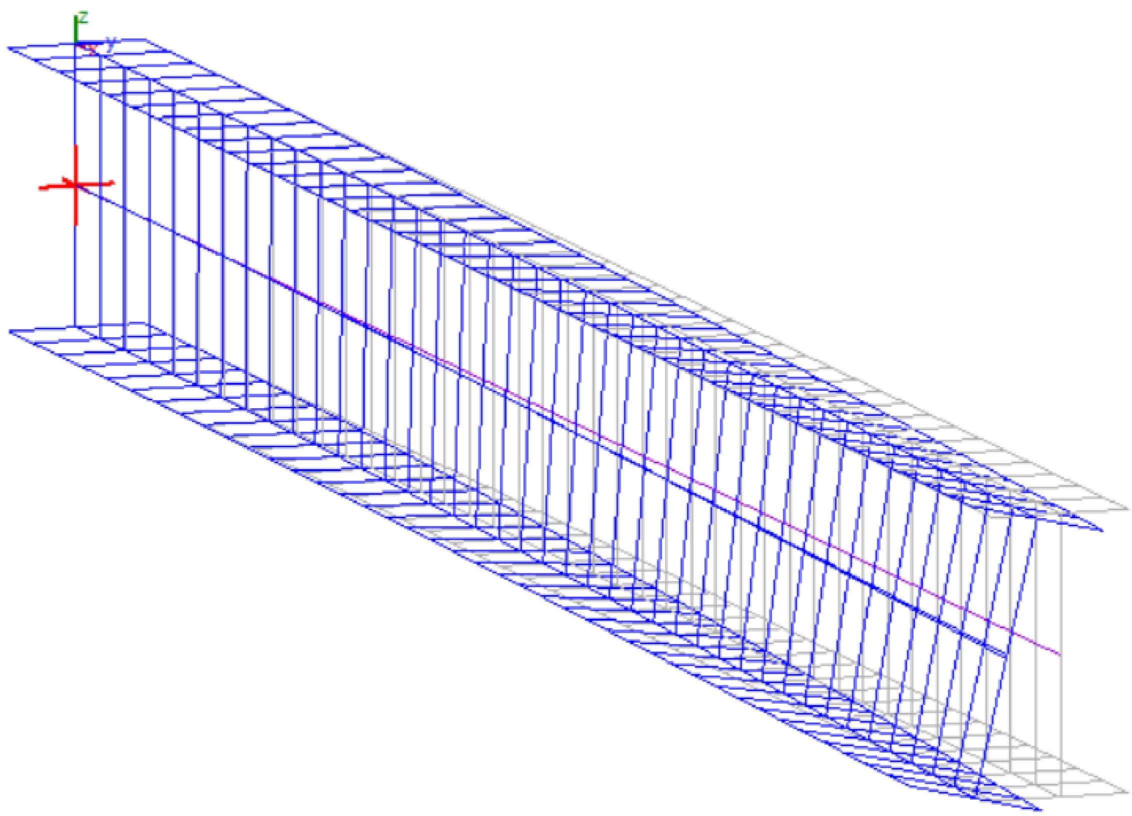


Figure 7: Cantilever with uniform moment, compression at bottom (LTBeamN)

3.3 Tertiary behaviour

There is another second order twist that we have not included. As the cantilever tip deflects laterally, the tip shear load will move out of the plane of bending and induce a torque about the root equal to the lateral displacement times the applied force. This torque would induce a further displacement of the beam in the same direction as the primary flange displacement. (Fig. 8)

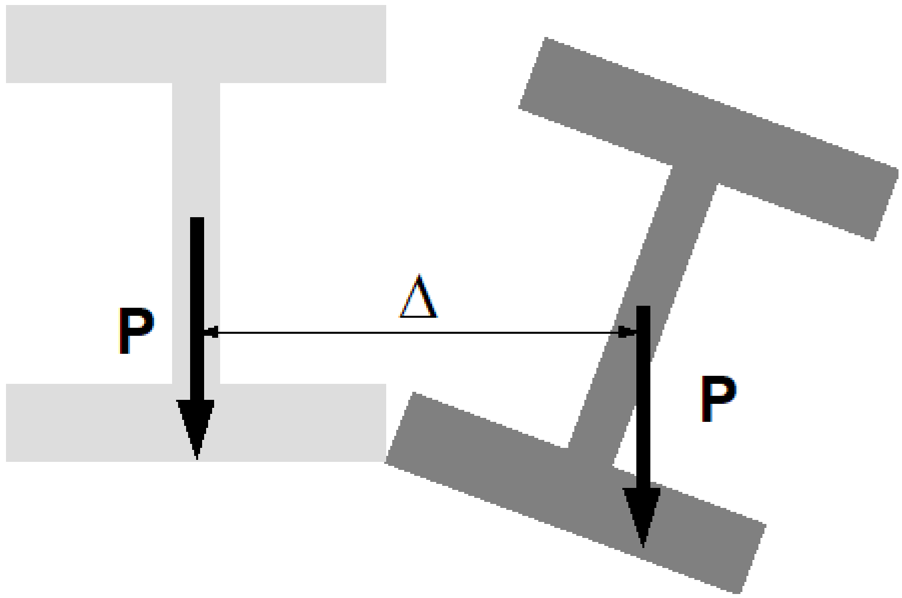


Figure 8: Tertiary torque resulting from lateral bending

In the case of the cantilever, this torque would induce a further displacement of the beam in the same direction as the tension flange displacement. The torque is not the cause for the general behaviour of the beam and the tendency of the tension flange to move further, as the initial displacement to one side is required for this torque to form. If the compression flange had the larger displacement, this torque would act against the twist at the flange tip. This phenomenon had been investigated for lateral motion of girders that were braced or restrained torsionally but allowed to move laterally. However, that work has not yet been reported. (White 2018)

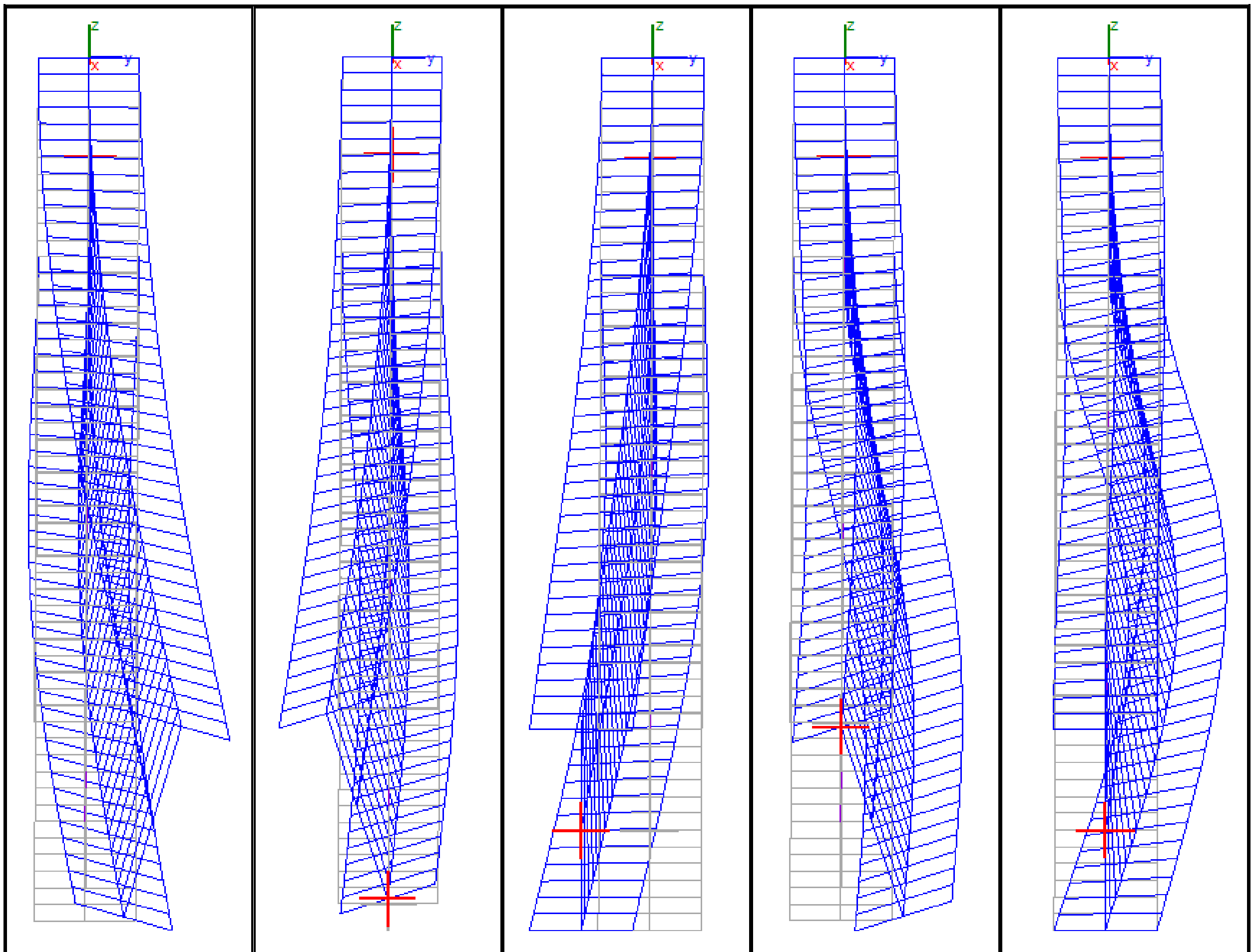
4. Bracing and restraint of cantilevers

The displacement of the flanges of cantilever beams is of interest from an academic view, but also provides a need to differentiate the bracing requirements of cantilevers from those of doubly supported beams. There are numerous works that fill this need, so they will only be mentioned briefly here, with the recognition that the different behaviour does need to be treated properly.

4.1 Reaction to restraints

When the cantilever is restrained from its "normal" movement, the beam's deflection changes, in some cases drastically. Restraining the lateral movement of the two flanges independently gives different responses.

When the tension flange is restrained laterally, the buckling response becomes very similar to the simply supported beams. This constraint also significantly increases the load at buckling.



Multiplier 1.0	Multiplier 1.10	Multiplier 1.56	Multiplier 1.96	Multiplier 2.48
Tip unrestrained	Compression flange restrained	Tip rotation restrained	Tension flange restrained	Tip rotation and translation restrained

Figure 9: Effect of restraining cantilever tip displacement IPE 300, 5.0 m long

The responses also include elevated critical buckling moment values according to the “Multiplier” number noted in Fig. 9. However, it must be noted that these increases will vary with different factors, and the values presented here are only for the cross section and length modelled.

4.2 Critical Flange

The Australian steel design standard AS4100 (Standards Australia, 2020) recognises that neither the compression flange nor the tension flange will always have the larger lateral displacement and uses a concept of the “critical flange” when considering placement of bracing when designing beam bracing. The critical flange is defined as the flange that, without restraint, would deflect to most. As this is the flange that will be braced in the structure, the note about “without restraint” is required. For beams (or beam segments) that are braced at both ends, the critical flange is the one in compression. If only one end is restrained, then the critical flange is the tension flange. (Clause 5.5 AS4100)

4.3 Bracing provisions

It is important that the flange that deflects furthest be braced laterally. (Yura 2001) Restricting the smaller displacement is not efficient. That restriction could also restrict a displacement that is opposite in sense to the displacement in the greater direction. This would induce an extra torque on the beam and push the cross section further out of vertical, worsening the condition that we intend to improve.

AS4100 (Standards Australia 2020) in Clause 5.4 notes that the critical flange shall be prevented from lateral translation, or the cross section prevented from rotation at that section. The section elided from the following quote are references to figures in the standard.

> 5.4.2.1 Fully restrained
>
> A cross-section of a member may be assumed to be fully restrained if either –
>
> (a) the restraint or support effectively prevents lateral deflection of the critical flange […], and effectively prevents twist rotation of the section […]; or partially prevents twist of the section […]; or
>
> (b) the restraint or support effectively prevents lateral deflection of some other point in the cross-section, and effectively prevents twist rotation of the section […].

The AISC Specification (AISC, 2022) requires bracing (Appendix 6, Clause 6.3 (1)) quoted below:

> Lateral bracing shall be attached at or near the beam compression flange, except as follows:
>
> (a) At the free end of a cantilevered beam, lateral bracing shall be attached at or near the top (tension) flange.
>
> (b) For braced beams subjected to double curvature bending, bracing shall be attached at or near both flanges at the braced point nearest the inflection point.
>
> It is permitted to use either panel or point bracing to provide lateral bracing for beams.

Torsional bracing is also permitted per Clause 6.3 (2).

In the Canadian design standard, S16's (CSA (2024)) general specifications (Clause 9.3) it is noted that lateral bracing shall be applied to both flanges at cantilever tips and at inflection points. Provisions for torsional bracing is less specific. The following is quoted from S16:24:

> 9.3 [...] Bracing for beams shall provide lateral restraint to the compression flange, except that at cantilevered ends of beams and beams subject to double curvature, the restraint shall be provided at both top and bottom flanges unless otherwise accounted for in the design.
>
> 9.4 Twisting and lateral displacements
>
> Twisting and lateral displacements shall be prevented at the supports of a member or element unless accounted for in the design.

Yura (2001) notes that torsional bracing can be less sensitive to position than lateral bracing. The use of bracing to restrain rotation can be used in place of lateral restraints. However, S16 does not include guidance on design of torsional bracing. The addition of guidance on torsional bracing to

S16 would be helpful to Canadian engineers, giving them greater options and allowance of more effective bracing techniques.

5. Summary

Lateral torsional buckling is a full section beam instability driven by applied moments, not solely by the region of compression in the beam. The instability is exhibited through the displacements of cross-section elements (primarily flanges) and the stresses they carry, as those reflect the influence of the applied moment's lateral and torsional components and the interaction of those components.

Classical beam buckling, with a uniform moment on a simply supported beam as the basis of the interaction, produces a common behaviour, though one that is sometimes misunderstood. As the compression flange tends to deflect more in the direction it originally deflects, the cross-section moves generally in that direction during buckling, causing the compression flange to move further.

In the case of a cantilever beam, where the moment is typically driven from a point load at the tip, or a uniformly distributed load, the flange that displaces further is the tension flange (typically the “top” flange in gravity loaded cantilevers). This can be explained in terms of the mechanism by which the moment components interact. For this set of boundary conditions, the flange behaviour with respect to lateral motion is for the compression flange to “push out” from the cantilever root and produces a lateral shear that moves the flange towards its original orientation. The tension in the other flange “pulls back” toward the root, producing a lateral shear that pushes the tension flange farther from its original orientation. These shears combined with the lateral torsional buckling interaction give the tension flange a larger displacement.

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