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A novel bracing concept for structural stability of ideal plane frames

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Abstract
This study focuses on the stability analysis of ideal plane braced frames with members subjected only to axial forces, and on determining the minimum bracing stiffness required to ensure, whenever possible, full bracing of the braced structures. Through the analysis of some examples, we show the limitations of the concept of full bracing and present a fundamental condition for the existence of full bracing solutions. To address the cases where full bracing is not possible, an alternative novel bracing concept is proposed and implemented in order to obtain the minimum bracing stiffness for which the critical buckling mode resembles, as much as intended, a non-sway buckling mode. To perform the stability analyses required for this study, a computational methodology based on the Singular Value Decomposition of the structures' global stiffness matrix is implemented and validated on structures with previously published results. Finally, this methodology is used to obtain minimum bracing solutions for two model frames.

1. Introduction

To perform the stability analyses required for this work, it is necessary to address a phenomenon that may occur in slender structures subjected to sufficiently large compression forces, known as buckling.

To fix ideas, let us consider the structure in Fig. 1: a two-storey unbraced plane frame with slender columns such that, under sufficiently large compressive conservative forces applied at the nodes, the frame deforms to a (buckled) configuration other than the reference (straight or undeformed) configuration before the material starts to respond nonlinearly. For sufficiently slender columns, this buckling process is initiated and occurs exclusively within the framework of Hooke's law. The phenomenon we will be looking at is the elastic buckling of columns or frames, occurring when the straight configuration of compressed columns ceases to be stable.

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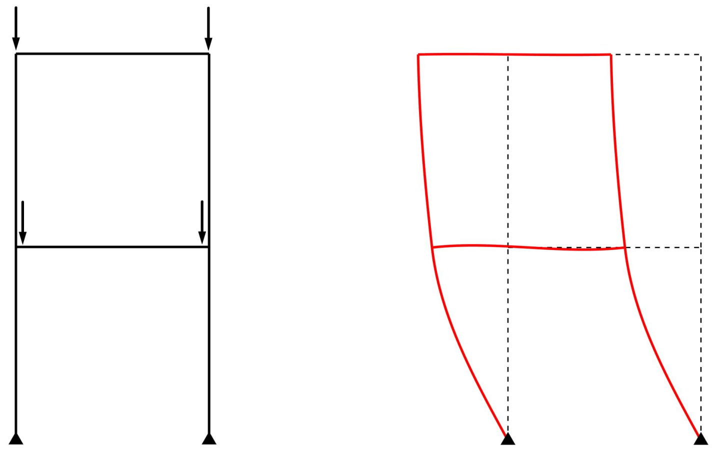

Figure 1: Undeformed (left) and buckled (right) configurations of a two-storey unbraced frame subjected to uniform compression in the columns.

As represented in Fig. 1, a slender unbraced frame subjected to sufficiently large compressive forces tends to a buckled configuration with arbitrarily large lateral displacements at floor levels: a sway buckling mode.

To prevent the structure from buckling into such a configuration, a bracing system may be added to the structure (Fig. 2, on the left). As the stiffness of the bracing system is gradually increased, the buckling mode's lateral displacements at the braced levels gradually reduce, until (under certain conditions) the stiffness of the bracing system reaches a finite threshold value above which the braced frame only buckles without lateral displacements at the braced nodes: a non-sway buckling mode (Fig. 2, on the right).

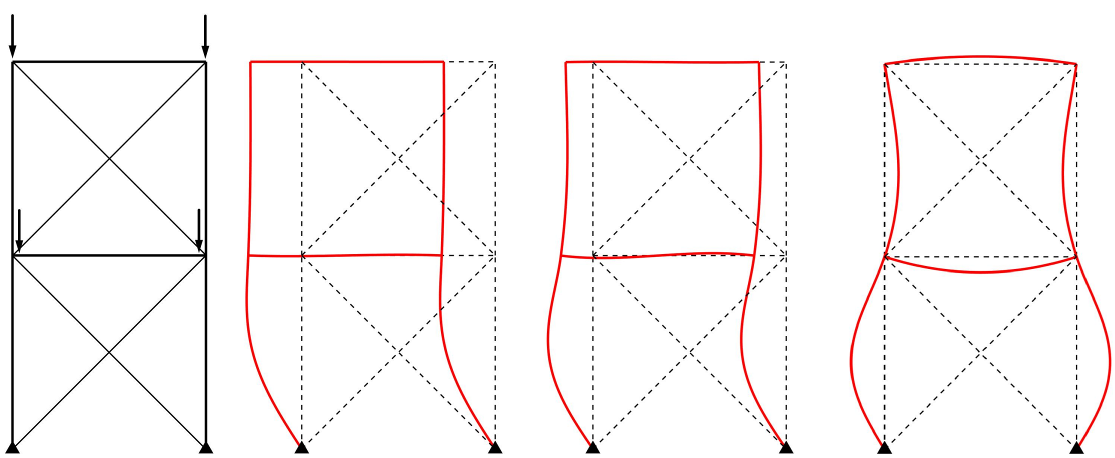

Figure 2: Evolution of the first buckling mode of the braced version of the frame in Fig. 1, as the stiffness of the bracing system increases. A non-sway first buckling mode is eventually reached (full bracing situation).

When the buckling mode of a braced frame assumes a non-sway configuration, then full bracing has been achieved. The threshold bracing stiffness above which full bracing occurs is known as "ideal" bracing stiffness, K_{id} (Winter 1960). Since this work not only addresses full bracing but also introduces a novel bracing concept, we will refer to this threshold bracing stiffness simply as "minimum" bracing stiffness, $K_{\min}$.

The present work derives from the MSc dissertation entitled "*Target Bracing of Slender Plane Frames: A generalisation of the concept of full bracing*" (Rosa 2024), which was originally intended to be solely focused on full bracing. Since full bracing proved to be a concept of very limited applicability, it became clear that it was necessary to generalise this bracing concept and allow minimum bracing stiffness solutions to be obtained for a larger set of braced structures. This necessity resulted in the proposal of a novel bracing concept, named "target bracing", that relaxes the intrinsic zero lateral displacement condition associated with full bracing, and replaces it with a different target defined by the designer. In order to achieve a generalisation of the concept of full bracing, the proposed concept of target bracing was associated with a "Target Displacement Ratio Criterion" (TDRC) that ensures, for the critical buckling mode, that the quotient between the braced nodes' maximum lateral displacement and the braced structure's maximum lateral displacement is equal to, or less than, a (non-vanishing, but small) limit prescribed by the designer. Pairing target bracing with a TDRC leads to a minimum bracing stiffness for which the buckling mode resembles, as much as intended, a non-sway buckling mode. In this presentation, the term "non-sway" is always used to define a buckling mode without any lateral displacements at the braced nodes.

This work focuses on the key aspects in (Rosa 2024). The structures considered in this work are ideal plane braced frames that, in addition to the bracing elements, are connected to classical bilateral supports such as clamped (or fixed) supports, pinned supports, sliding supports, or sliding-clamped supports. The loading on the structures consists of vertical concentrated nodal forces aligned with the central axes of the columns, and the axial deformability of the columns is considered negligible; therefore, the frames' columns will only be subjected to axial forces. It is assumed that the structural elements remain elastic until the onset of buckling. All beams and columns are considered rigidly connected, prismatic, and orthogonal to themselves. The bracing elements are assumed to be sufficiently slender to not support vertical loads, to have negligible compressive resistance, and to work only in tension. Additionally, the stiffness of the bracing systems is controlled by a single stiffness variable, K.

It is also important to note that a "real" system is a set of springs coupled in series - for example, a connection, "plus" a brace, "plus" a connection - and, as such, the stiffness of the bracing system is necessarily smaller than the stiffness of the brace itself. To be more precise, the stiffness of the bracing system is smaller than the smallest stiffness in the series association of stiffnesses. In this work, when we use the term "bracing stiffness", we are referring to the stiffness of the bracing system as a whole (the brace "plus" the connections).

2. The limitations of full bracing

This chapter will be focused on illustrating the limitations associated with full bracing. Published examples of full bracing solutions can be found in (Winter 1960, Olhoff and Akesson 1991, Yang 1991, Zhang et al. 1993, Ziemian and Ziemian 2017, Ziemian and Ziemian 2021, Rosa 2024).

2.1 Cantilever column subjected to uniform compression

To illustrate the limitations of full bracing, let us consider the case of a prismatic axially rigid cantilever column with a horizontal elastic support at the top of the column (Fig. 3). The column has length L, flexural stiffness EI, and a compressive concentrated load P applied at the top of the column.

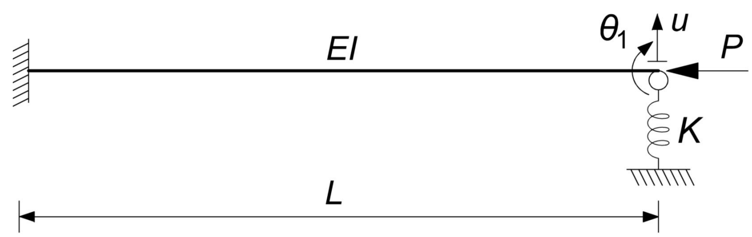

Figure 3: Cantilever column with a lateral elastic support and a concentrated axial load at the top.

For the column in Fig. 3, the system of equilibrium equations can be written as

$$\begin{pmatrix} 12\phi_2(\lambda)\cdot\lambda\cot\lambda + k & 6\phi_2(\lambda) \\ 6\phi_2(\lambda) & 3\phi_2(\lambda)+\lambda\cot\lambda \end{pmatrix}\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{1}$$

where $k = KL^3/EI$ represents the non-dimensional bracing stiffness, $d_1 = u/L$, $d_2 = \theta_1$, $\phi_2(\lambda) = \lambda^2/[3\cdot(1\text{-}\lambda\cot\lambda)]$ is one of the stability functions (Livesley and Chandler 1956), and $\lambda = (L/2)\cdot(P/EI_c)^{0.5}$ is the non-dimensional load parameter of the stability functions.

Full bracing is achieved if there is a finite threshold bracing stiffness (K_{min}) above which an increase in the bracing stiffness does not affect the critical load. If so, the first buckling mode of the braced column with $K = K_{min}$ will be a non-sway buckling mode ($d_1 = 0$). Thus, by considering $d_1 = 0$, which corresponds to eliminating the first column of the stiffness matrix in Eq. 1, we can obtain the non-sway system of equilibrium equations

$$\begin{pmatrix} 6\phi_2(\lambda) \\ 3\phi_2(\lambda)+\lambda\cot\lambda \end{pmatrix}(d_2) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2}$$

For Eq. 2 to have a non-trivial solution, there should be a real positive value of the non-dimensional load parameter λ for which both $\phi_2(\lambda) = 0$ and $3\phi_2(\lambda) + \lambda\cot\lambda = 0$, which is impossible. Therefore, there is no value of λ for which $d_1 = 0$, and full bracing is impossible to achieve for this example. An analysis of this example is also included in (Olhoff and Akesson 1991). For further context relating to this example refer to (Rosa 2024).

2.2 Cases lacking a full bracing solution

As we just showed, even an ideal structure as simple as a cantilever column is not guaranteed to have full bracing solutions. In this regard, there are far more cases of ideal structures for which full bracing is impossible than cases where full bracing is possible. Moreover, if imperfections are taken into account (real structures), then full bracing is never possible. Fig. 4 shows some examples of ideal frames where full bracing is impossible (Rosa, 2024).

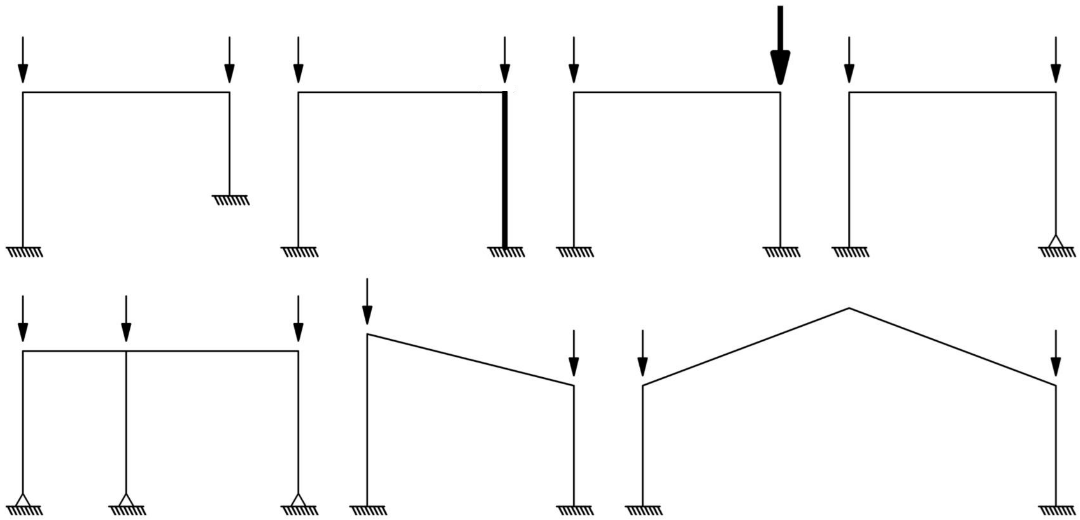

Figure 4: Examples of ideal frames without full bracing solutions.

Aside from ideal structures where full bracing is never possible, there are also ideal structures for which full bracing is only achievable if braced at specific locations. To clarify this statement, let us evaluate how three bracing configurations affect the bifurcation loads of a simply supported prismatic column under uniform compression (Fig. 5).

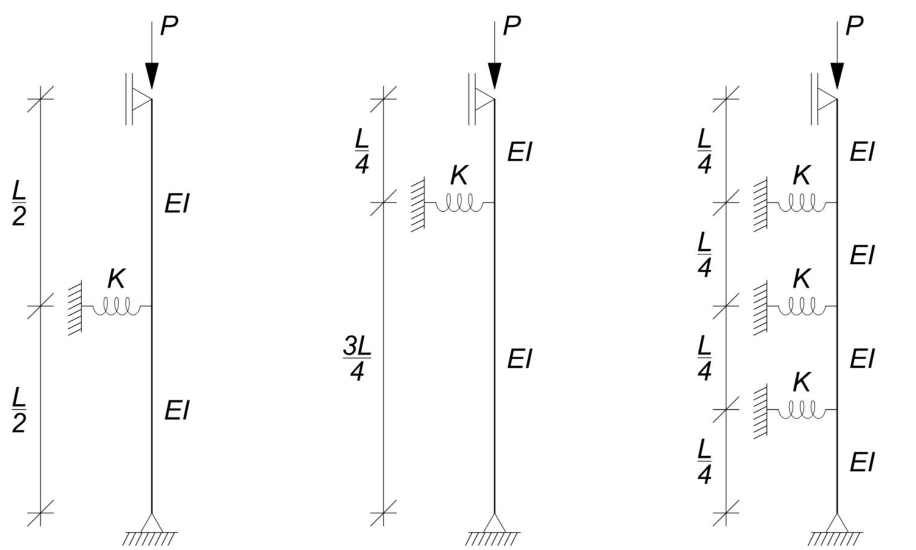

Figure 5: Simply supported prismatic column under uniform compression with three different bracing configurations: one brace at mid-height (left), one brace at a distance of 0.75L from the base of the column (center), and three equally spaced and equally stiff braces along the length of the column (right).

Before we proceed, it will also be important to recall how a simply supported prismatic column buckles under uniform compression (the Euler column). Under uniform axial compressive stress, the buckling mode of order n of the Euler column is made of n sinusoidal semi-waves of constant length, L/n (Fig. 6). As such, the buckled configuration of the n^{th} buckling mode has $n-1$ intersections with the undeformed configuration of the column at well-defined locations.

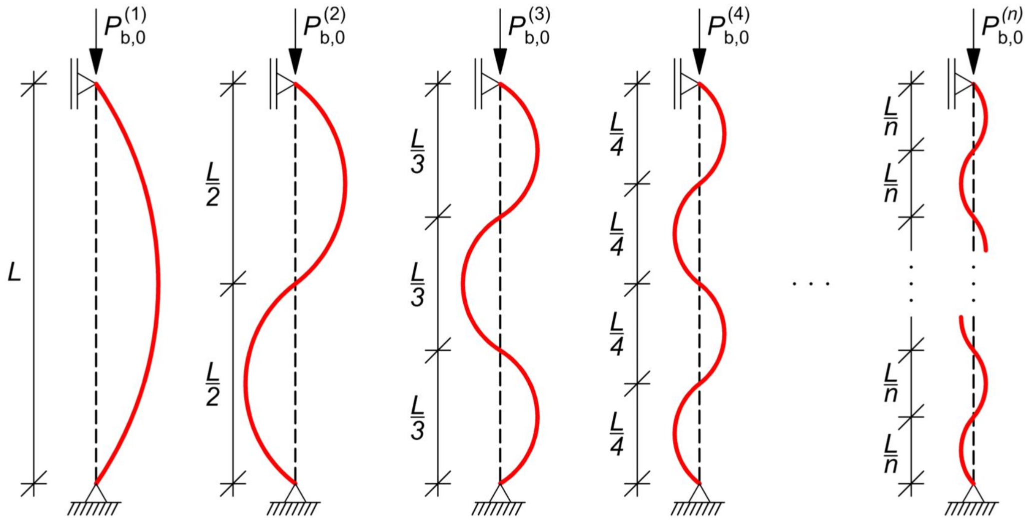

Figure 6: Schematic representation of the buckling mode shapes of the uniform Euler column.

Fig. 7, Fig. 8 and Fig. 9 present, for each of the bracing configurations in Fig. 5, how the first six bifurcation loads of the column (represented by the non-dimensional parameter $\lambda_{bif}^{(n)}$, where n = 1, 2, 3, 4, 5, 6) evolve with the increase of the bracing stiffness (represented by the non-dimensional parameter k).

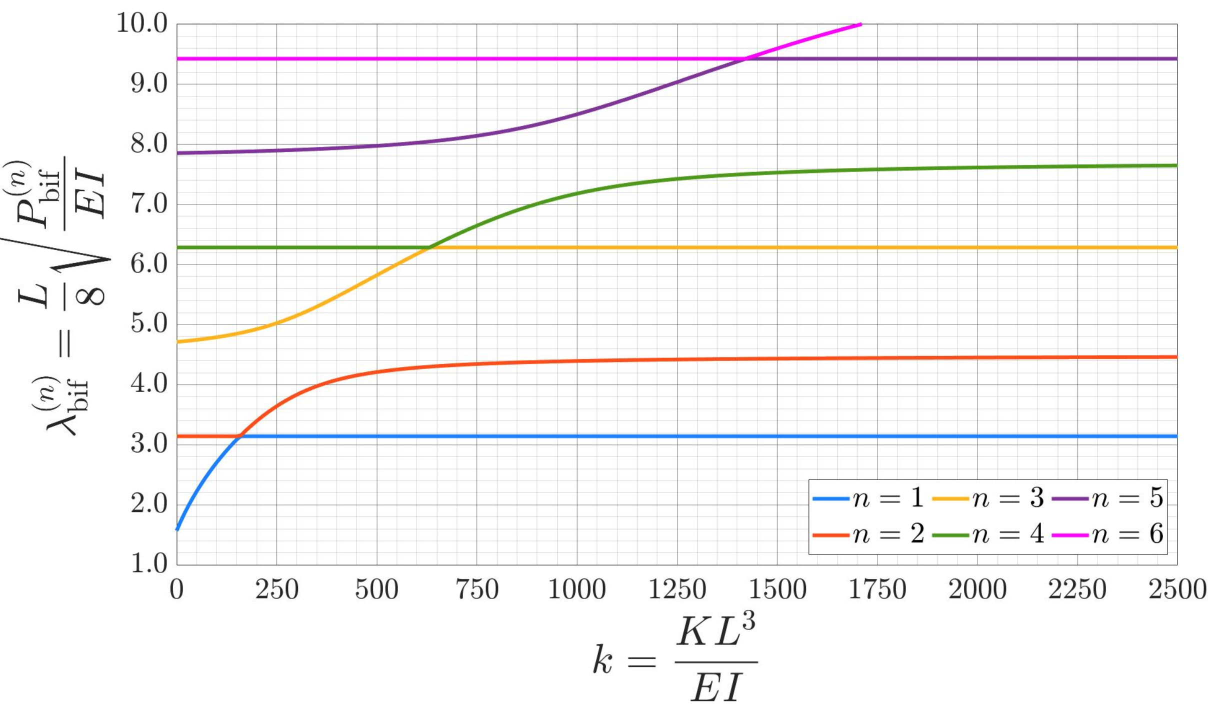

Figure 7: Evolution of the first six non-dimensional bifurcation load parameters $\lambda_{bif}^{(n)}$ (n = 1, 2, 3, 4, 5, 6) with the increase of the non-dimensional bracing stiffness parameter k for the case of one brace placed at a distance of 0.50L from the base of the column (Fig. 5, on the left).

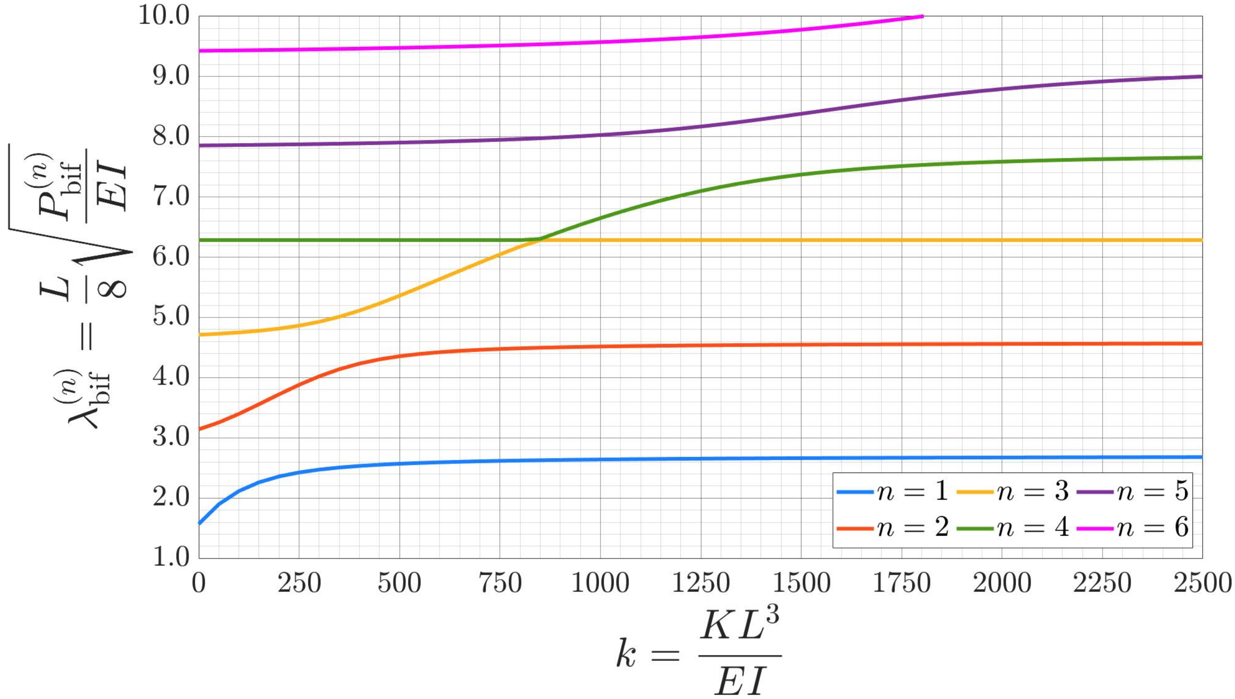

Figure 8: Evolution of the first six non-dimensional bifurcation load parameters $\lambda_{\mathrm{bif}}^{(n)}$ (for $n = 1, 2, 3, 4, 5, 6$) with the increase of the non-dimensional bracing stiffness parameter k for the case of one brace placed at a distance of $0.75L$ from the base of the column (Fig. 5, center).

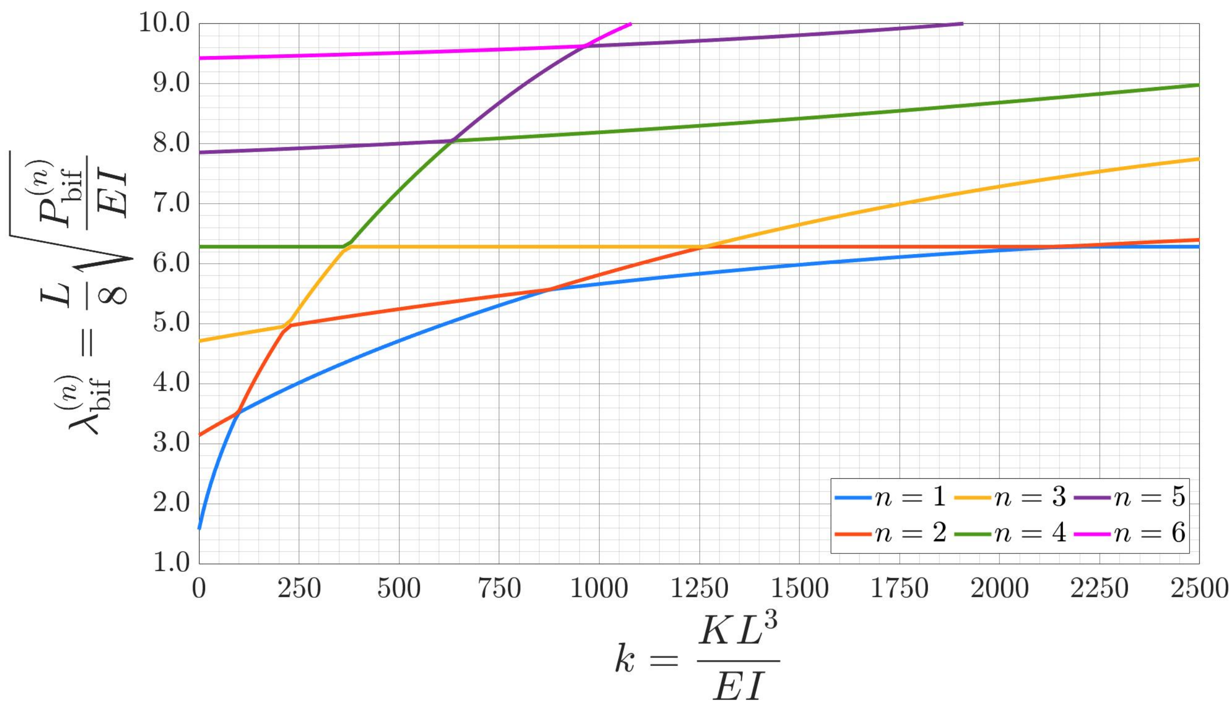

Figure 9: Evolution of the first six non-dimensional bifurcation load parameters $\lambda_{\mathrm{bif}}^{(n)}$ (for $n = 1, 2, 3, 4, 5, 6$) with the increase of the non-dimensional bracing stiffness parameter k for the case of three equally stiff braces placed at distances of $0.25L$, $0.50L$ and $0.75L$ from the base of the column (right of Fig. 5, on the right).

All three sets of plots display bifurcation loads that are not affected by the increase of the bracing stiffness (horizontal lines). These unchanging values represent the bifurcation loads of the buckling modes without lateral displacements at the braced nodes (non-sway modes), and all three bracing configurations share a non-sway buckling mode, represented by the originally green line, which corresponds to the fourth buckling mode in Fig. 6.

On the other hand, the bifurcation loads of the buckling modes that have lateral displacements at any of the braced nodes are sensitive to the braces, and their values increase (non-linearly) with the increase of the bracing stiffness. Although more considerations can be made from the analysis of these plots, let us focus on the main point of this example.

Considering the plot in Fig. 8 and comparing it with the plots in Fig. 7 and Fig. 9, a key difference is apparent: there is not a value of the bracing stiffness for which the critical load (n = 1) becomes constant. In other words, in Fig. 8 the blue curve never reaches the lowest horizontal line (the non-sway critical load). Therefore, for the bracing configuration represented in the center of Fig. 5, full bracing of the column is impossible. However, as Fig. 7 and Fig. 9 indicate, a prismatic simply supported column subjected to uniform compression can be fully braced as long as all inflection points of a buckling mode higher than the first buckling mode (n > 1) are restrained with a sufficiently stiff bracing system.

3. General formulation of full bracing and target bracing

3.1 Formulation of full bracing

The fundamental condition that a structure must satisfy for full bracing to be possible is that the critical load of the rigidly braced structure ($K = \infty$, for all braces) must be equal to one of the bifurcation loads of the unbraced structure (K = 0, for all braces), which is mathematically represented by

$$\exists n \in \mathrm{N} : P^{(n)}_{\mathrm{bif},0} = P^{(1)}_{\mathrm{bif},\infty} (= P_{\mathrm{cr},\infty}). \tag{3}$$

Finding the bifurcation loads of both limit cases, K = 0 and $K = \infty$, leads to a classic eigenvalue and eigenvector problem. If the bifurcation loads returned by the eigenvalue analysis satisfy Eq. 3, then full bracing is possible and the minimum bracing stiffness required to achieve full bracing may be computed with the algorithm represented by the flowchart in Fig. 10.

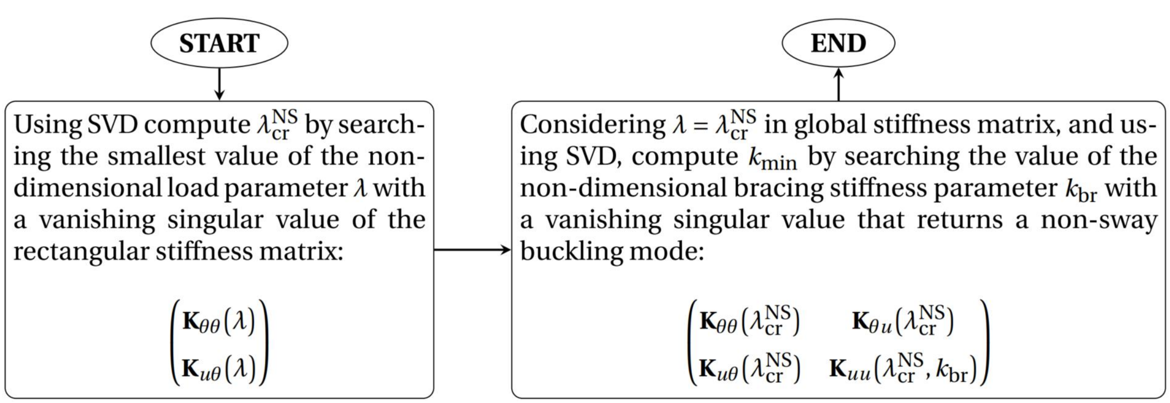

Figure 10: Flowchart of the algorithm implemented in (Rosa 2024) to find full bracing solutions.

The Singular Value Decomposition (SVD) of the rectangular matrix in the first step of the flowchart in Fig. 10 yields the non-dimensional non-sway critical load parameter λ_{cr}^{NS} (the first bifurcation load corresponding to a non-sway buckling mode).
If the flowchart in Fig. 10 is applied without validating the fundamental condition for the existence of a full bracing solution (Eq. 3), and without checking if the buckling mode that results from the second step of the flowchart is non-sway, *it may yield false full bracing solutions*: the minimum bracing stiffness returned by the algorithm in the flowchart may not in fact represent the threshold stiffness above which the critical load and buckling mode are non-sway, but rather represent the threshold stiffness above which the order number n of the first non-sway buckling mode remains constant at the lowest level it can be demoted to, and which may not correspond to $n = 1$ (the critical buckling mode). An example of this would be the previously analysed case of a simply supported prismatic column, subjected to uniform compression, and braced at a distance of $0.75L$ from the base of the column (center of Fig. 5), where the misapplication of the algorithm in Fig. 10 would (wrongly) return the intersection between the yellow and green lines in Fig. 8 as the minimum bracing stiffness solution.

3.2 Formulation of target bracing
As stated throughout this presentation and demonstrated in the previous chapter, full bracing is only applicable to an extremely limited set of structures. Furthermore, even for structures that have full bracing solutions, it is essential to ensure that the correct number of braces is correctly placed, and that a thorough stability analysis is performed in order to identify correct full bracing solutions, and avoid false solutions.

In order to define an alternative to full bracing that could be applied to any ideal structure, a more comprehensive bracing concept was devised and implemented for the first time, to the best of our knowledge, in (Rosa 2024), which was named "target bracing".

In (Rosa 2024) and in this presentation, this novel bracing concept is associated with a "Target Displacement Ratio Criterion" (TDRC) that relaxes the intrinsic zero lateral displacement condition of full bracing, and ensures that, for the critical buckling mode of the braced frame, the ratio between the maximum lateral displacement at the floor levels ($u_{i,\max}$) and the maximum lateral displacement recorded in the columns of the braced frame ($\Delta_{\max}$) is less than a sufficiently small value (e_d) specified by the designer. The flowchart in Fig. 11 illustrates the approach that was implemented to find the target bracing solutions presented in the next chapter.

Using this approach (target bracing "+" TDRC) it is possible to find, for any ideal structure, a threshold value of the bracing stiffness ($K_{\min}$) that ensures that the critical buckling mode shape of the braced frame closely resembles a (perfect) non-sway buckling mode.

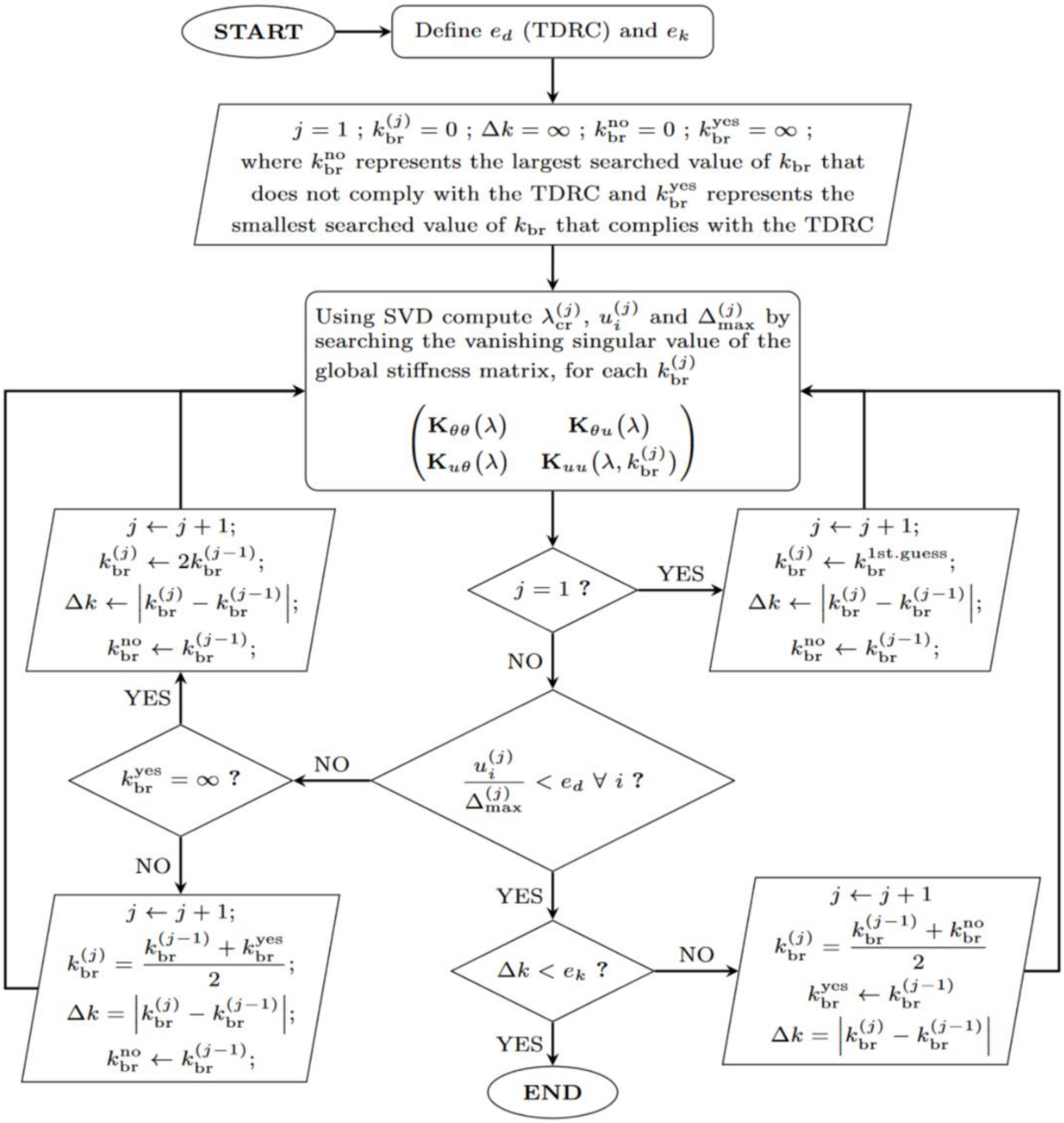

Figure 11: Flowchart of the algorithm proposed in (Rosa 2024) to find target bracing solutions.

3.3 Singular Value Decomposition

The stability analyses carried out in this work were performed through the implementation of the Singular Value Decomposition factorization method (Golub and van Loan 1996, Stewart 1998), which allows the determination of the structure of rectangular matrices. Our analyses were based on the vanishing of the lowest singular value of the structures' stiffness matrices to compute the bifurcation loads, buckling mode shapes, and minimum bracing stiffnesses.

3.4 Computational validation

In order to validate the computational methodologies implemented in this presentation, we analysed three test frames that were compared with already published results (Reis and Camotim 2012); this validation was presented in (Rosa 2024).

4. Numerical examples

4.1 Full bracing

The structure that is considered to show the implementation of the full bracing algorithm (Fig. 10) is the case of a one-storey symmetric braced frame with fixed supports (represented in Fig. 12). The frame has height h, span s and two downward concentrated loads P applied at the top of the columns. Both columns have cross sections with flexural stiffness EI_c, are considered to be axially rigid ($EA_c = \infty$), and are subjected to a uniform compression force P. The beam has cross sections with flexural stiffness EI_b, and is also assumed to be axially rigid.

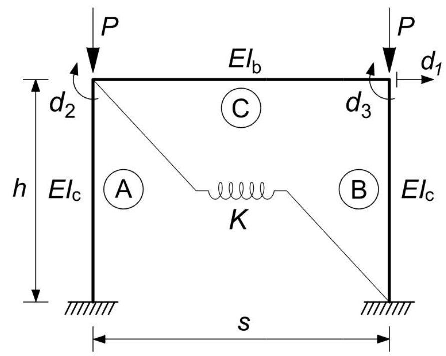

Figure 12: One-storey symmetric frame with fixed supports and concentrated loads at the top of the columns.

Considering the stability functions of each structural element of the frame, and the independent displacements represented in Fig. 12, we get the exact global stiffness matrix of the braced frame

$$\overline{\mathrm{K}} = \begin{pmatrix} 24\phi_1(\lambda)+k & 6\phi_2(\lambda) & 6\phi_2(\lambda) \\ 6\phi_2(\lambda) & 4\phi_3(\lambda)+4g & 2g \\ 6\phi_2(\lambda) & 2g & 4\phi_3(\lambda)+4g \end{pmatrix}, \tag{4}$$

where $g = (h/s)\cdot(I_b/I_c)$, $\lambda = (h/2)\cdot(P/EI_c)^{0.5}$, and $k = (Kh^3/EI_c)$ are non-dimensional parameters associated, respectively, with geometric properties of the frame and its members,, the effect of the axial load on the elements' stiffness, and the bracing stiffness; $\phi_1(\lambda) = \phi_2(\lambda)\cdot\lambda\cot\lambda$, $\phi_2(\lambda) = \lambda^2/[3\cdot(1-\lambda\cot\lambda)]$, and $\phi_3(\lambda) = 0.75\phi_2(\lambda) + 0.25\lambda\cot\lambda$ (Livesley and Chandler 1956, Reis and Camotim 2012).

The first step in the analysis of a full bracing problem is to check the fundamental condition for the existence of a full bracing solution (recall Eq. 3). To do so, it is necessary to compute the bifurcation loads of the frame in Fig. 12 for $K = 0$ and for $K = \infty$.

Fig. 13 shows the non-dimensional bifurcation load parameters of the frame in Fig. 12 for $K = 0$ (red continuous curves) and $K = \infty$ (black dashed curves). These results show that the critical load of the rigidly braced frame (lowest black dashed curve) matches the second bifurcation load of the unbraced frame (second lowest red continuous curve), and confirms that the frame in Fig. 12 fulfills Eq. 3 for every plotted value of the non-dimensional geometric parameter g. It is worth noting that, although the values are very similar, the third bifurcation load of the unbraced frame and the second bifurcation load of the rigidly braced frame in Fig. 13 do not match for any value of the non-dimensional geometric parameter (Rosa, 2024).

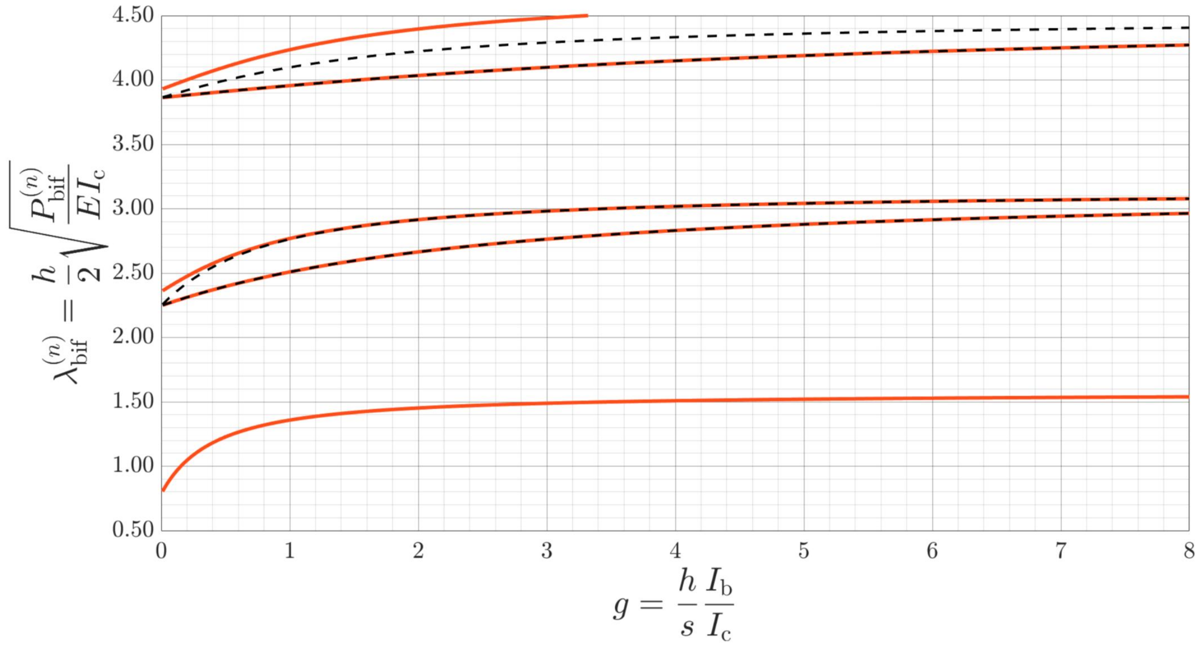

Figure 13: Graph of the first five non-dimensional bifurcation load parameters of the unbraced frame ($K = 0$; red continuous curves), and of the first four non-dimensional bifurcation load parameters of the rigidly braced frame ($K = \infty$; black dashed curves).

Having confirmed the validity of Eq. 3, we can apply the full bracing algorithm in Fig. 10. Fig. 14 presents the solutions for the minimum non-dimensional bracing stiffness parameter, k_{min}, obtained from the computational implementation of the full bracing algorithm.

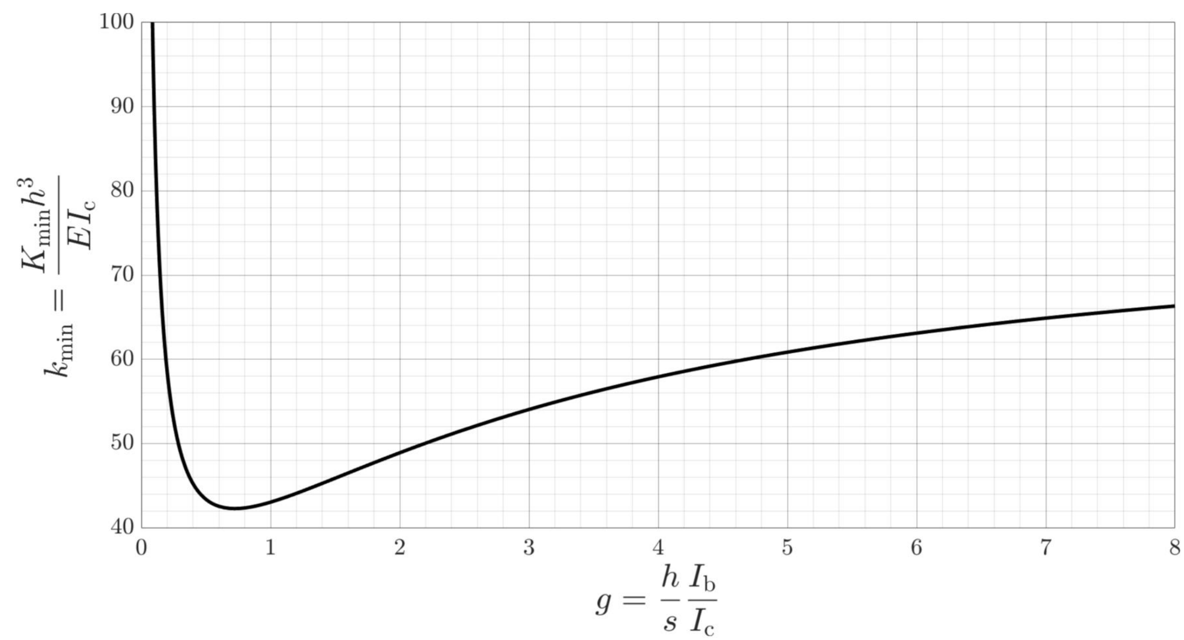

Figure 14: Graph of the non-dimensional minimum bracing stiffness parameter k_{min} as a function of the non-dimensional geometric parameter g of the frame in Fig. 12.

4.2 Target bracing

To illustrate the target bracing algorithm (Fig. 11), we consider the case of a one-storey frame with fixed supports and asymmetric concentrated loads (P, αP) at the top of the columns, represented in Fig. 15.

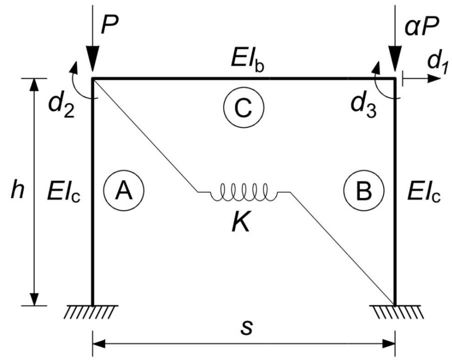

Figure 15: One-storey symmetric frame with fixed supports and asymmetric concentrated loads (P, αP) at the top of the columns.

Unlike full bracing problems, target bracing problems do not require any previous condition to be validated in order to proceed with the stability analysis, so we will proceed directly to the minimum non-dimensional bracing stiffness solution obtained from the flowchart in Fig. 11. Fig. 16 shows the minimum non-dimensional bracing stiffness parameter solutions $k_{\min}$ obtained from the target bracing flowchart for a TDRC of 1%, α = {0, 0.2, 0.4, 0.6, 0.8, 0.96, 0.98, 0.999}, and g between 0 and 8. For reference, the solutions for $\alpha = 1$ (that corresponds to full bracing) are also included in Fig. 16 (black dashed curve).

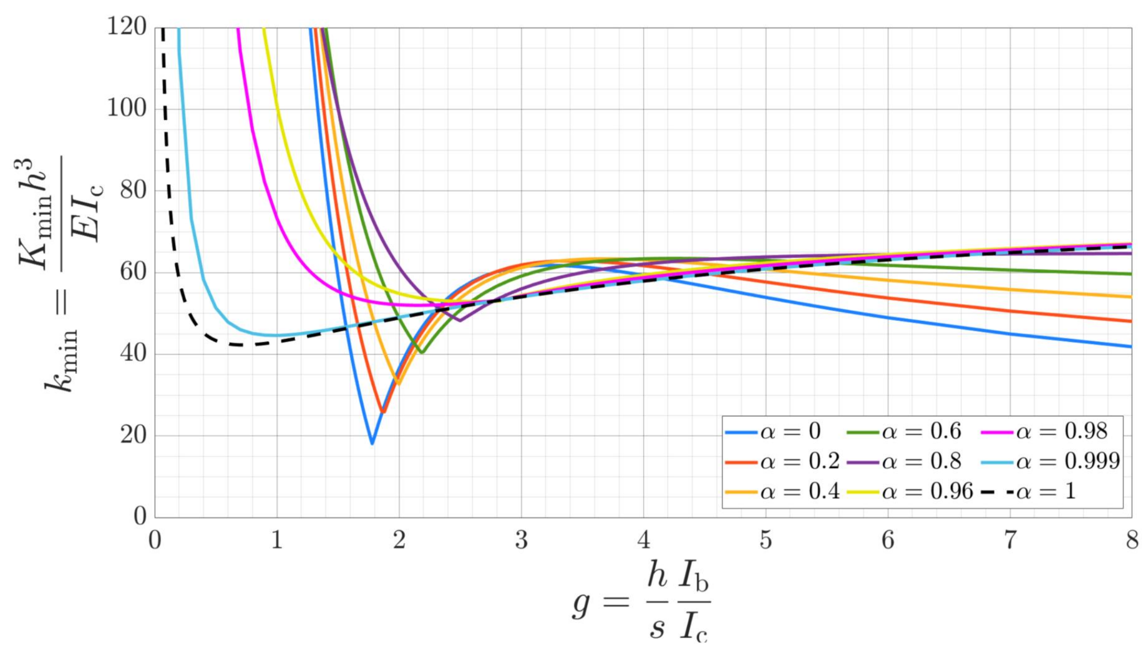

Figure 16: Graph of the non-dimensional minimum bracing stiffness parameter $k_{\min}$ as a function of the non-dimensional geometric parameter g of the frame with asymmetric concentrated loads (P, αP) in Fig. 15.

The curves in Fig. 16 display a relatively similar behavior to the full bracing solution (α = 1), with two clear distinctions: i) the curves have an angular point with a sharp global minimum (as if they resulted from the intersection of two smooth curves); and ii) the curves also have a local maximum before trending asymptotically as g tends to infinity. The results also show that as the load asymmetry vanishes (as α tends to 1), the minimum bracing stiffness solution curves tend to the full bracing solution.

5. Conclusions

In this work, it was shown that the concept of full bracing is of limited applicability as a result of the fundamental condition for the existence of full bracing solutions, and a novel alternative bracing concept named "target bracing" was proposed and implemented.

The concept of target bracing was illustrated considering a Target Displacement Ratio Criterion (TDRC) of 1% in order to find minimal bracing stiffness solutions that correspond to a buckling mode that closely approximates a perfect non-sway buckling mode. The combination of target bracing with a strict TDRC may be considered a "*quasi*-full bracing".

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