



A design method to calculate the flexural buckling critical force of columns with partially continuous elastic support

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Abstract

In this paper, the global flexural buckling of columns with lateral elastic supports is discussed. In practice, lateral supports are provided by components attached to the column, e.g., a gypsum board attached to a wall stud through mechanical fasteners. If the fasteners are (approximately) equally spaced, a reasonable mechanical model for the problem is a ‘column on elastic foundation’, from which the critical force can be calculated by simple analytical formulae. However, if there is a ‘gap’ in the lateral support, i.e., only one portion of the length of the column is laterally supported, while the rest remains unsupported, this can influence the buckling mode and critical force – this influence is discussed here. First, a semi-analytical model is developed to solve the buckling problem exactly. The model is a column with eccentric elastic point supports, i.e., springs. The springs can work in any lateral direction or against rotation. The displacements of the column are assumed as linear combinations of trigonometric functions, e.g., one or multiple sinewaves for pinned–pinned supports. This model provides the exact critical forces. Based on the observed tendencies of the critical forces in the presence of partial quasi-continuous lateral support, a simple calculation method is proposed that considers the effect of unsupported length. The proposed model is simple enough to be employed in design, and parametric numerical studies suggest that its accuracy satisfies design expectations. Though this paper discusses flexural buckling only, the method can be extended to torsional and flexural-torsional buckling, which will be the subject of future work.

1. Introduction

Particularly in light steel framing columns are frequently supported by attached building components. A typical case is a cold-formed steel wall stud, supported by sheathing, see Fig.1 (left), and/or further components such as ledger beams, resilient channels, and bridging (not shown in Fig. 1). The stud’s connections to the attached components are realized by fasteners, that is, the stud is laterally supported at discrete locations. Obviously, these lateral supports are far from being rigid. A proper mechanical model for the stud, therefore, is a column supported by discrete springs, and under service conditions, the springs can be assumed to be elastic. The spring rigidity depends on the details of the attached building component and the fastener. In the case of cold-formed steel

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studs, spring stiffness values are available for many practical cases, see, e.g., Schafer (2013), Schafer (2020), and AISI S100 (2024).

The primary action on the stud is compression, hence the stud is subjected to instabilities, including global, distortional, and local buckling. Global buckling consists of either flexural buckling, torsional buckling, or flexural-torsional buckling, depending on the cross-section shape and orientation. In this paper, the focus of the work is on minor-axis flexural buckling, i.e. only the buckling that is in the plane of the wall.

For light steel framing, the sheathing is typically the most efficient attached component against flexural buckling in the plane of the wall. In a typical case, the sheathing-to-stud fasteners are spaced equally (at least approximately), hence the lateral support is ‘quasi-continuous’, a reasonable mechanical model in this case is a ‘column on elastic foundation’. If the stud is laterally supported along its whole length, the critical force can be calculated by simple analytical formulae. However, if there is a ‘gap’ in the quasi-continuous lateral support (as illustrated in Fig. 1 – and would be common in ledger framing for light steel framed walls), i.e., only one portion of the stud is laterally supported while the rest remains unsupported; this can influence the buckling mode and critical force. This influence is the focus of this paper.

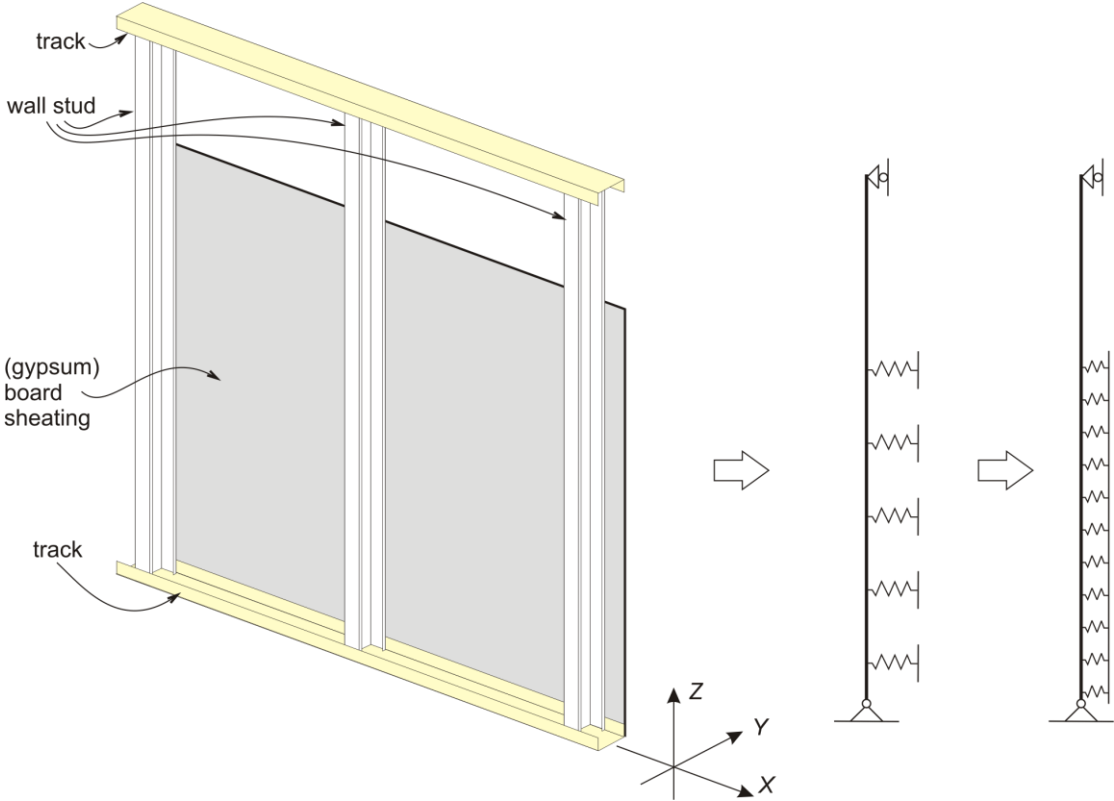


Figure 1: A sample wall panel and its mechanical models

First, a semi-analytical model is developed to solve the buckling problem exactly. The model is a column with eccentric elastic point supports, i.e., springs, briefly summarized in Section 2. Then, in Section 3, studies are presented on columns with full quasi-continuous lateral supports, showing that an equivalent buckling length can readily be introduced either numerically or analytically.

Then, columns with partial lateral quasi-continuous supports are investigated numerically, exploring the effect of the length of the unsupported part (as well as of a few other parameters); the results showing the equivalent buckling lengths as presented in Section 4. Based on the results, a simple approximate method is proposed in Section 5 for the calculation of the (lowest) critical force for flexural buckling of columns with partially continuous lateral elastic supports. The accuracy of the method is studied by comparing the exact and approximate critical forces.

2. Semi-analytical model for columns with discrete elastic supports

A semi-analytical ‘column-with-springs’ method is developed and applied here for the ‘exact’ calculation of the critical force. The details of the method are provided in Schafer (2020) and are briefly summarized as follows.

The column’s mechanical model is one-dimensional (i.e., a beam model), therefore, the column is assumed to be an assembly of infinitely many rigid cross-sections, characterized by the section properties. Elastic springs can be added to any cross-section. Three types of springs are considered: transverse springs parallel with the x axis (k_{x1} , k_{x2} , etc.), transverse springs parallel with the y axis (k_{y1} , k_{y2} , etc.), and rotational springs activated by the twisting rotation of the column ($k_{\theta1}$, $k_{\theta2}$, etc.). The springs’ locations are arbitrary within the cross-section. The model is illustrated in Fig. 2. In this paper, the column ends are assumed to be pinned.

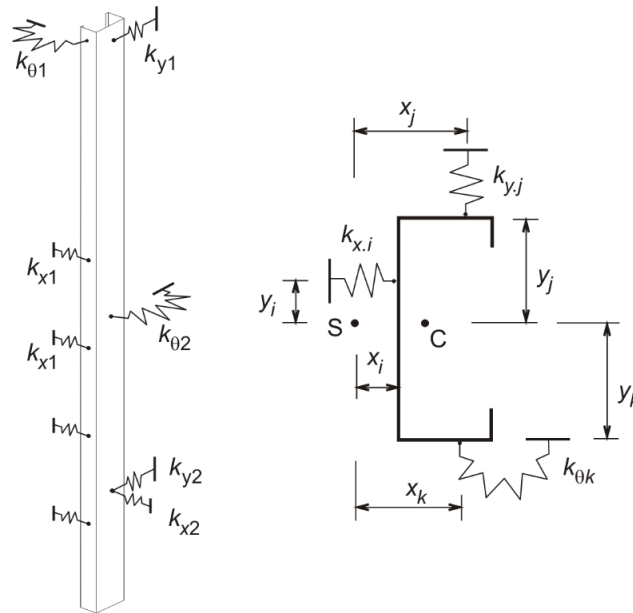


Figure 2: ‘Column-with-springs’ model

The calculation follows the steps of the classic analytical derivation for the column buckling, based on the energy method: the total potential is expressed and minimized, however, in the internal potential (i.e., the elastic strain energy) the energy accumulated in the attached discrete springs is considered, too. The cross-sections’ displacements have three components: translation along the x -axis, translation along the y -axis, and twisting rotation. Due to the presence of lateral elastic supports, the longitudinal distributions of the displacements are general, therefore, they are expressed by the sum of multiple sinusoidal shape functions. Accordingly, the degrees of freedom

(DOF) of the problem are the amplitudes of the sinusoidal terms. If the problem is simplified to a few DOF (e.g., flexural-torsional buckling problem with a single sine term for each displacement component, or, e.g., flexural buckling problem with a few sine terms for one of the translational components), the problem can be solved analytically and the critical forces can be expressed in closed form. As the DOF number increases, a closed-form expression for the critical forces becomes impractical and/or impossible. At the same time, the total DOF number needed to solve a typical column problem remains small, hence the problem can easily be solved numerically. In the current research, MatLab (The MathWorks Inc., 2022) software has been employed for the numerical solutions.

3. Equivalent length – full continuous elastic support

3.1 Derivation

First, let us consider a column with discrete springs. The springs are equally spaced, along the full length of the column. Flexural buckling only is discussed here, the buckling is assumed parallel with the x -axis, accordingly, x -directional springs only are present. The discrete springs can be converted to an elastic foundation (i.e., distributed spring), as:

$$\bar{k}_x = \frac{k_x}{s_s} \quad (1)$$

where k_x is the discrete spring stiffness, and s_s is the spring spacing. The buckling of a column on elastic foundation is a classic stability problem discussed in textbooks (Timoshenko and Gere, 1961), the critical force, in general, is expressed as follows:

$$P_{cr.a}^{spring} = \frac{\pi^2 EI_y}{L_{hw.y}^2} + \frac{\bar{k}_x L_{hw.y}^2}{\pi^2} \quad (2)$$

where I_y is the minor-axis moment of inertia of the cross-section, E is the Young's modulus, $L_{hw.y}$ is the half-wave length of the buckled shape. The lowest critical force depends on the parameters, and can be expressed (with some approximation) as:

$$P_{cr.a.min}^{spring} = 2\sqrt{\bar{k}_x EI_y} \quad (3)$$

Now, it is possible to assign an $L_{eq.a}$ equivalent (buckling) length from the equivalence of critical forces, as follows:

$$\frac{\pi^2 EI_y}{L_{eq.a}^2} = 2\sqrt{\bar{k}_x EI_y} \quad (4)$$

$$L_{eq.a} = \pi \left(\frac{EI_y}{4\bar{k}_x} \right)^{0.25} \quad (5)$$

where $L_{eq.a}$ would be utilized for L in the classical $P_{cr} = \pi^2 EI_y / L^2$ formula for flexural buckling.

3.2 Numerical results

For the numerical studies an 8-ft-long simply supported (stud) column is employed. Two standard cold-formed steel lipped channels are considered: a ‘thin’ (or ‘light’) one (600S162-43) and a ‘thick’ (or ‘heavy’) one (600S300-97). Equally spaced discrete springs are used, representing screw fasteners, the screw distance being a variable. Three values of spring stiffnesses are considered as follows: for the ‘thin’ column the spring stiffnesses are 0.1, 0.5, and 2.5 kips/in; for the ‘thick’ column the stiffness values are 0.6, 3.0, and 15.0 kips/in. It is to note that 0.5 and 3.0 kips/in can be considered as typical values provided by gypsum boards; accordingly, the considered spring stiffnesses are intended to represent ‘weak’, ‘typical’, and ‘strong’ spring supports.

Two equivalent lengths (L_{eq}) are calculated for each case. One comes from the analytical model, i.e., Eq. (5). The other one is derived from the semi-analytical method (described in Section 2), using the equivalence as follows:

$$\frac{\pi^2 EI_y}{L_{eq,s}^2} = P_{cr,s.min}^{spring} \quad (6)$$

$$L_{eq,s} = \pi \left(\frac{EI_y}{P_{cr,s.min}^{spring}} \right)^{0.5} \quad (7)$$

where $P_{cr,s.min}^{spring}$ is the lowest critical force (to the investigated flexural mode) from the semi-analytical ‘column-with-springs’ method.

The comparison of equivalent lengths is presented in Tables 1 and 2 for the ‘thin’ and ‘thick’ columns, respectively. Noticeable differences occur only if the equivalent length is around 40-55 inches when Eq. (5) slightly overestimates L_{eq} (which means underestimation of the critical force). This is due to the approximate nature of the analytical formula. (It is to note that the analytical formula is ‘exact’ only if the number of actual buckling halfwaves is large enough; this condition is violated in the actual cases unless the spring spacing is very small and/or the spring stiffness is large.) Nevertheless, the results suggest that the ‘column on elastic foundation’ model can be used with satisfactory accuracy for columns with equally spaced discrete lateral springs.

Table 1: Comparison of critical forces: ‘thin’ columns

screw spacing (in)	‘weak’ support			‘typical’ support			‘strong’ support		
	Eq. (7) (in)	Eq. (5) (in)	ratio (5)/(7)	Eq. (7) (in)	Eq. (5) (in)	ratio (5)/(7)	Eq. (7) (in)	Eq. (5) (in)	ratio (5)/(7)
2	32.0	32.1	1.00	21.4	21.5	1.00	14.3	14.4	1.00
4	37.7	38.2	1.01	25.2	25.5	1.01	17.1	17.1	1.00
6	40.3	42.3	1.05	27.3	28.3	1.03	18.7	18.9	1.01
8	41.9	45.4	1.08	30.0	30.4	1.01	20.1	20.3	1.01
10	42.8	47.5	1.11	31.7	31.8	1.00	21.2	21.3	1.00
12	46.1	50.3	1.09	33.6	33.6	1.00	22.5	22.5	1.00
14	48.5	52.0	1.07	34.7	34.7	1.00	23.3	23.2	1.00
16	51.3	54.0	1.05	36.0	36.1	1.00	24.1	24.2	1.00
18	54.7	56.5	1.03	37.4	37.8	1.01	25.1	25.3	1.01
20	54.7	56.5	1.03	37.4	37.8	1.01	25.1	25.3	1.01
22	58.8	59.8	1.02	39.0	40.0	1.03	26.6	26.7	1.01
24	58.8	59.8	1.02	39.0	40.0	1.03	26.6	26.7	1.01

Table 2: Comparison of critical forces: ‘thick’ columns

screw spacing (in)	‘weak’ support			‘typical’ support			‘strong’ support		
	Eq. (7) (in)	Eq. (5) (in)	ratio (5)/(7)	Eq. (7) (in)	Eq. (5) (in)	ratio (5)/(7)	Eq. (7) (in)	Eq. (5) (in)	ratio (5)/(7)
2	36.2	36.3	1.00	24.2	24.3	1.01	16.2	16.2	1.00
4	40.8	43.2	1.06	28.2	28.9	1.03	19.0	19.3	1.02
6	42.9	47.8	1.12	31.9	32.0	1.00	21.3	21.4	1.00
8	47.7	51.4	1.08	34.3	34.4	1.00	23.0	23.0	1.00
10	51.0	53.8	1.05	35.8	36.0	1.00	24.0	24.0	1.00
12	55.1	56.9	1.03	37.5	38.0	1.01	25.1	25.4	1.01
14	57.6	58.8	1.02	38.5	39.3	1.02	25.7	26.3	1.02
16	60.4	61.1	1.01	39.5	40.9	1.03	26.4	27.3	1.03
18	63.7	63.9	1.00	40.6	42.8	1.05	27.9	28.6	1.03
20	63.7	63.9	1.00	40.6	42.8	1.05	27.9	28.6	1.03
22	67.6	67.6	1.00	41.8	45.2	1.08	30.1	30.2	1.00
24	67.6	67.6	1.00	41.8	45.2	1.08	30.1	30.2	1.00

4. Equivalent length – partial continuous elastic support

In this section, columns with partial lateral supports are studied numerically, assuming that the springs are missing at the top part of the column (as might commonly occur in light steel framing when the floors are ledger framed), while both ends remain pinned. The length of the unsupported top portion, g , varies over a wide range, i.e. from 0 to 72 inches. (Note, the column length is 8 ft = 96 in.)

The calculated flexural equivalent lengths (L_{eq}) for the three spring stiffnesses (‘weak’, ‘typical’, ‘strong’) and two stud types (‘thin’/‘light’, ‘thick’/‘heavy’) are provided in Fig. 3. The most important observations are as follows.

When the gap length, g , is small, the equivalent length is nearly constant (and let us denote this as L_1). When the gap length is large, the equivalent length is linearly proportional to the gap length (and let us denote this as L_2). Between these two cases, there is a gradual transition. The transition depends on the spring stiffness: the stiffer the spring, the more abrupt the transition is. The L_1 equivalent length for small gaps is practically equal to the length that can be obtained from Eq (5) using the ‘column on elastic foundation’ model. The two straight lines (i.e., the horizontal one representing the equivalent length for small gaps, and the inclined one representing the equivalent length for large gaps) intersect at a certain gap value (and let us denote this as L_{g1}), which is approximately proportional to L_1 . The inclination of the inclined straight line is approximately constant for both column types and for all the considered spring stiffness values.

Note that in some cases the numerically determined L_{eq} is not smoothly changing with g , particularly when the spring spacing (s_s) is large, and especially when it is combined with a small unsupported length. In the numerical studies, the springs are equally spaced and there are springs at both the bottom and top ends of the supported part; accordingly, the provided s_s spacing value is ‘nominal’. The ‘real’ spacing can be different, and the difference between the ‘real’ and ‘nominal’ spacing is significant when g is small and s_s is large leading to the variation in results.

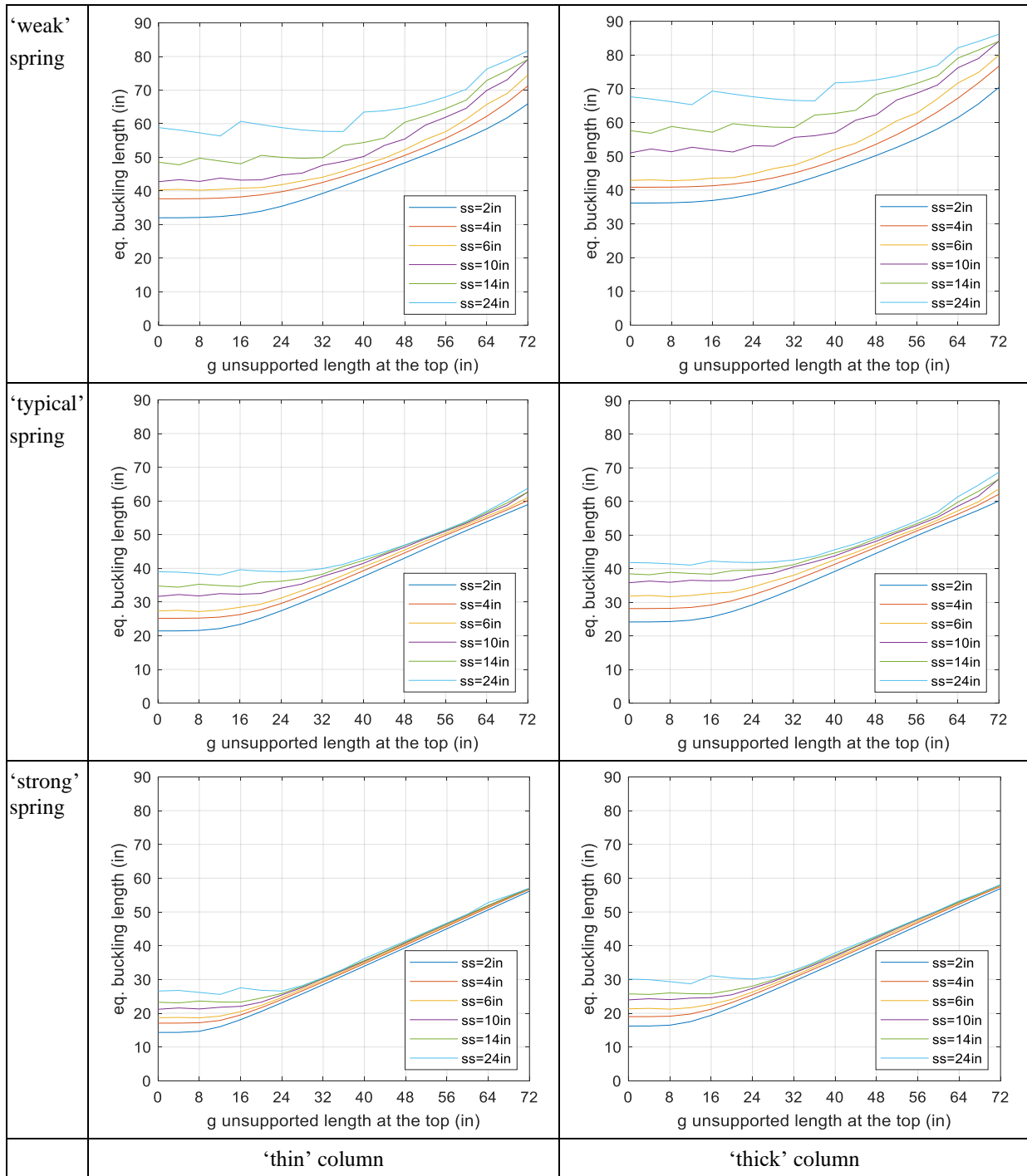


Figure 3: Equivalent buckling lengths calculated by the ‘column-with-springs’ method

5. Approximate method for columns with partial continuous elastic support

5.1 The approximate method

Based on the above observations, the equivalent buckling length, L_{eq} , may be provided in terms of the g unsupported length and can be approximated by two straight lines. One is a horizontal line, i.e., the equivalent length is constant and equal to the one derived from Eq. (5). This line is

governing when the unsupported length is small; hence, mechanically, the buckling happens as buckling of a column on an elastic foundation. The other line is inclined but approximately straight, meaning that the equivalent length is proportional to the unsupported length; hence, the buckling happens in the unsupported part of the column (of which the unsupported portion, however, is partially restrained at its ends).

As an approximation, therefore, the equivalent buckling length, L_{eq} , can be approximated as follows:

$$L_{eq} = \max(L_1, L_2) \leq L_{col} \quad (8)$$

where L_{col} is the column length (note, the column is assumed to be pinned at both ends), while L_1 is calculated from Eq. (5):

$$L_1 = \pi \left(\frac{EI_y}{4k_x} \right)^{0.25} \quad (9)$$

and L_2 is calculated as follows:

$$L_2 = L_1(1 - \alpha_1\alpha_2) + \alpha_1g \quad (10)$$

where g is the unsupported length, and α_1 and α_2 are constants dependent on the parameters of the problem. Coefficients α_1 and α_2 have easy-to-understand meanings, α_1 being the ratio of L_{g1} and L_1 , while α_2 is the tangent of the inclined straight line. Finally, the approximate value of the critical force can be calculated by the well-known Euler formula:

$$P_{cr}^{approx} = \frac{\pi^2 EI_y}{L_{eq}^2} \quad (11)$$

Note, the expression could also be used to generate an effective length factor, K , where:

$$K = L_{eq}/L_{col} \quad (12)$$

and

$$P_{cr}^{approx} = \frac{\pi^2 EI_y}{(KL_{col})^2} \quad (13)$$

5.2 Comparison of exact and approximate equivalent lengths

Based on the results presented in Fig. 4, α_2 is approximately constant for all the investigated cases and considered to be 0.65. α_1 is dependent on the spring stiffness, and can be taken as 0.6, 0.75, and 0.8 for ‘weak’, ‘typical’, and ‘strong’ springs, respectively.

Using these α_1 and α_2 values, the approximate equivalent lengths are calculated, using Eqs. (8-10), and the results are compared to the exact values calculated by the ‘column-with-spring’ method. The comparison is shown in Fig. 5 for three spring stiffnesses and two studs.

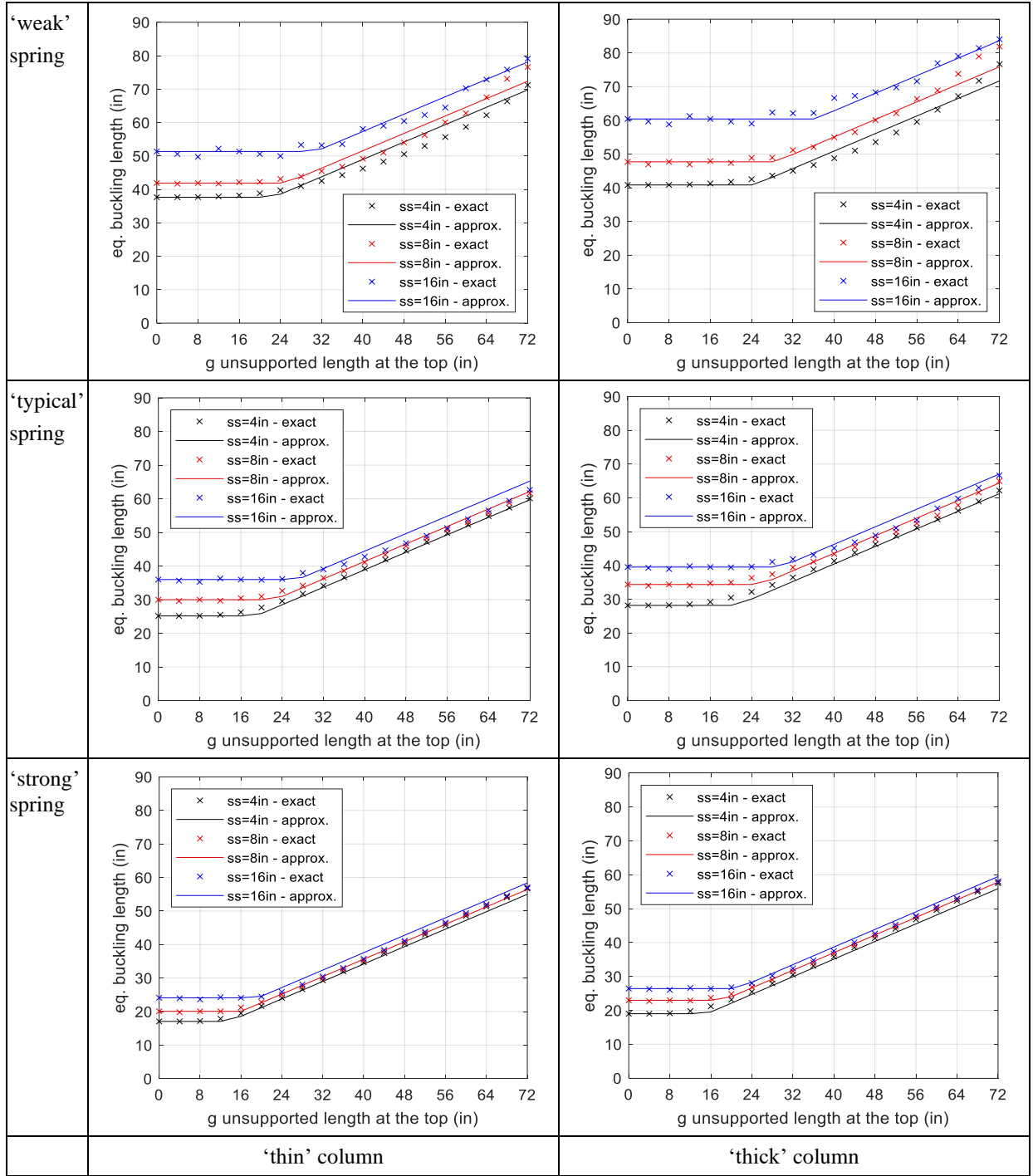


Figure 4: Exact and approximate equivalent buckling lengths

5.3 Comparison of exact and approximate critical forces

Once the equivalent length value is available, the approximate critical force can readily be calculated. In Fig. 5, the $P_{cr}^{approx} / P_{cr}^{exact}$ values are visualized, where P_{cr}^{exact} is the lowest critical force (to the investigated flexural mode) from the semi-analytical ‘column-with-springs’ method, while P_{cr}^{approx} is the result from Eq. (11).

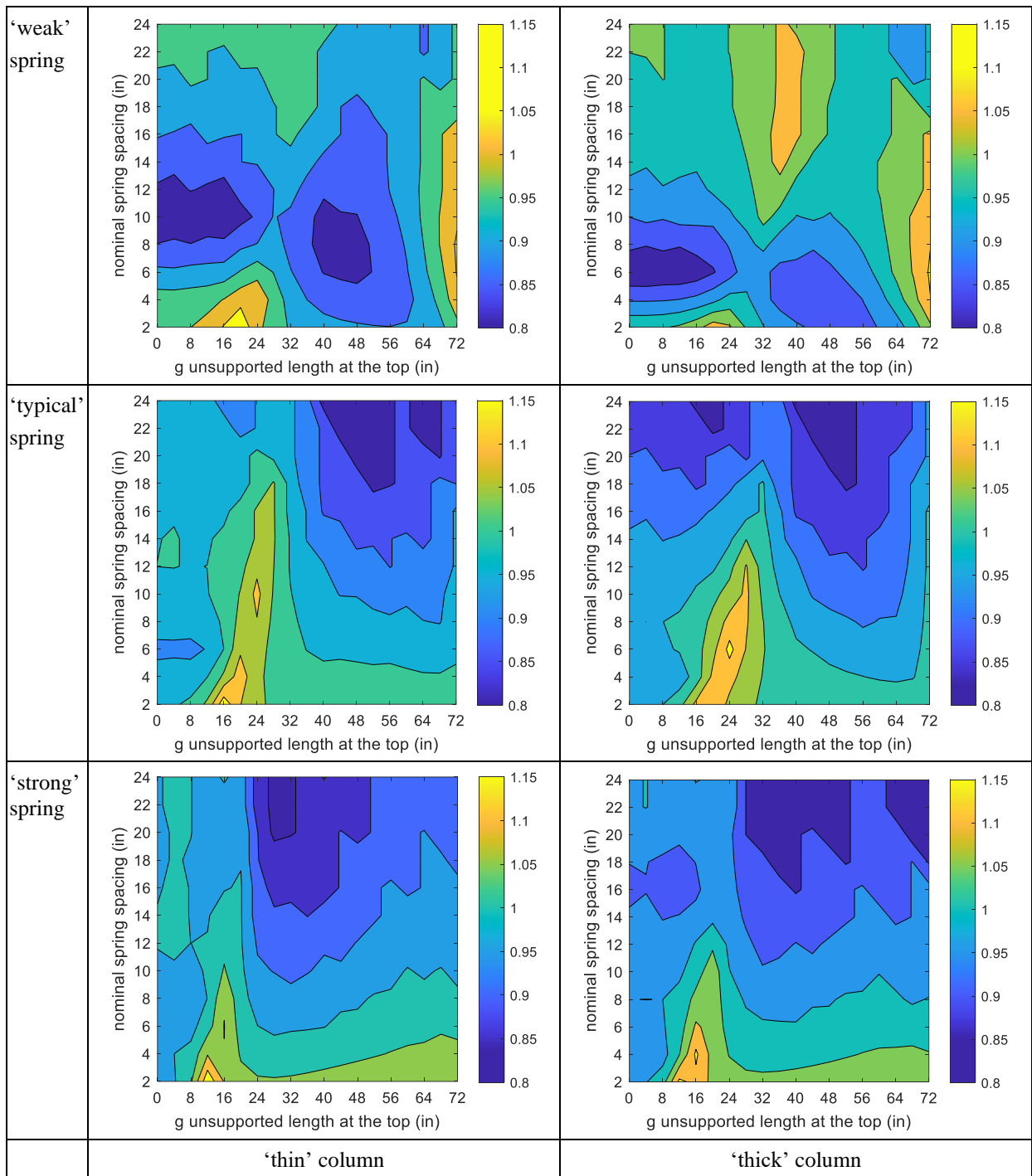


Figure 5: Approximate-to-exact critical force ratios

The contour plots of Fig. 5 suggest that the $P_{cr}^{approx}/P_{cr}^{exact}$ ratio is between 0.8 and 1.15 for the considered cases. If the ratio is smaller than 1, the critical force prediction is conservative, otherwise unconservative. Obviously, the approximate critical force values are controlled by the selection of α_1 and α_2 values; for the values employed here the relatively large differences between the approximate and exact critical forces (and particularly the unconservative results) are

limited to small areas. Statistical data for the practically more relevant spring spacing and gap values are summarized in Table 3. The critical force approximation is, in average, slightly conservative, with mean value below 1, and the coefficient of variation (CoV) is low, between 3% and 7%. Thus, the introduced approximate method provides a reasonable prediction.

Table 3: Statistical characteristics of approximate-to-exact critical force ratios: for s_s between 8 and 16 in, g between 24 and 48 in

	‘thin’/‘light’			‘thick’/‘heavy’		
	‘weak’ support	‘typical’ support	‘strong’ support	‘weak’ support	‘typical’ support	‘strong’ support
min	0.836	0.886	0.879	0.905	0.879	0.895
max	0.971	1.114	0.991	1.071	1.117	1.018
mean	0.887	0.985	0.937	0.984	0.988	0.956
CoV	0.034	0.062	0.031	0.044	0.067	0.031

6. Conclusions

In this paper, the flexural buckling of columns with quasi-continuous lateral elastic supports was discussed, focusing on those cases when the lateral support is partial, i.e., not present along the full length of the column. The quasi-continuous supports are due to discrete springs, equally spaced, where a spring represents the effect of a (typically non-structural) component attached to the column (e.g., a gypsum board connected to the stud by screws).

A semi-analytical method was employed to calculate the exact value of critical forces. The results suggest that the behavior is governed by one of the following three cases. (a) If the unsupported length is small, the buckling behavior can be accurately predicted by using a ‘column on elastic foundation’ mechanical model to which an analytical solution exists. (b) If the unsupported length is large, the buckling occurs mainly in the unsupported part of the column; however, the buckling length is smaller than the unsupported length and depends on the problem parameters. (c) Between these two cases, there is a gradual transitional zone; the results suggest that the stiffer the springs are, the smaller this transition zone is; hence, practically, this transition zone is significant only if the springs are very weak.

A simple approximate method was introduced to approximate the equivalent buckling length, through which the critical force of the column can readily be estimated. The proposed method is simple, suitable even for everyday design calculations. Based on the completed – though limited – numerical studies, the approximate method can predict the critical force value with approx. 10% accuracy. Occasionally, larger differences exist up to 20%, but the unconservative predictions are rare and limited to specific combinations of the parameters.

Future work is desirable as follows. Further numerical studies are necessary to test the introduced method on cases with different parameters. This can lead to defining application boundaries and/or introducing modifications. Moreover, though not discussed in this paper, the proposed approximate method can be extended to torsional and flexural-torsional buckling of columns with partial elastic lateral support.

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